Chapter 4 CURVES

Force Fields

According to Newton's laws of motion, a particle will move in a straight line at constant velocity unless it is subjected to forces. In that case it will accelerate according to Newton's third law

$$\mathbf{F} = m\mathbf{a} \tag{4.1}$$

where m is the mass of the particle. In this chapter we shall study the motion of particles subjected to variable forces. That is, we must allow the possibility that the force applied to the particle depends upon its position (as in gravitation) or even upon time (in the case of a variable electromagnet). This gives rise to the notion of a **field** of force. A field of force will be given in this way: at time t and position $\mathbf{x}$ a particle of unit mass will experience a force $\mathbf{F}(\mathbf{x}, t)$. Thus for each t_0 we have associated a vector $\mathbf{F}(\mathbf{x}, t_0)$ to each point $\mathbf{x}$. We can illustrate this as in Figure 4.1. Now, we have seen that a particle of unit mass situated at $\mathbf{x}_0$ at time $t = 0$, with a velocity $\mathbf{v}_0$ at $t = 0$ will follow the path of motion determined by the given field of force as the solution of the differential equation

$$\begin{aligned} \mathbf{f}''(t) &= \mathbf{F}(\mathbf{f}(t), t) \\ \mathbf{f}'(0) &= \mathbf{v}_0 \\ \mathbf{f}(0) &= \mathbf{x}_0 \end{aligned}$$

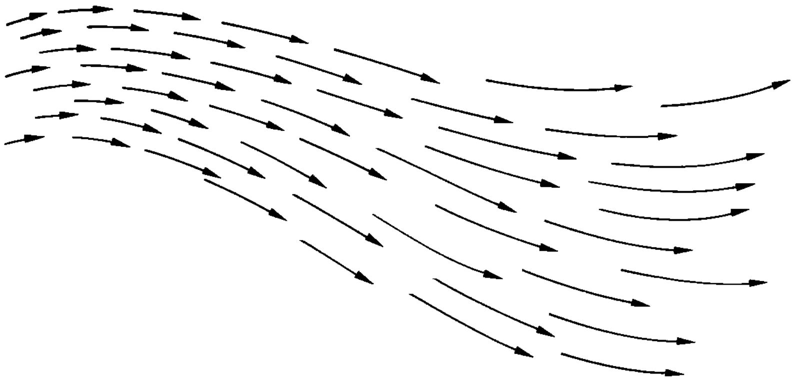


Figure 4.1

The path of motion is a curve in space given by the function $\mathbf{f}$ which solves this equation.

Examples

1. Suppose a particle moves around the unit circle in the plane according to the function

$$\mathbf{f}(t) = (\cos t, \sin t) \tag{4.2}$$

What force field would account for this motion? Differentiating twice we find that

$$\mathbf{f}'(t) = (-\sin t, \cos t)$$
$$\mathbf{f}''(t) = (-\cos t, -\sin t) = -\mathbf{f}(t)$$

Thus the particle is accelerating toward the origin with constant magnitude (see Figure 4.2). This motion can be accounted for by the force field

$$F(z, t) = -z$$

In fact, in the presence of this field, if a particle has a velocity at time $t = 0$ orthogonal to its position vector, then it will continue to move in a circle centered at the origin. We can see this by solving the differential equation

$$f''(t) = -f(t)$$
$$f(0) = z_0, f'(0) = iz_0$$

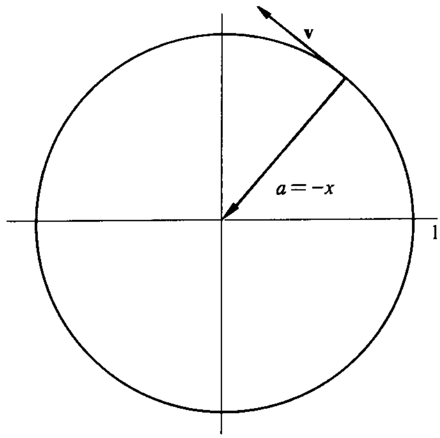


Figure 4.2

The solution of this equation is

$$f(z) = z_0 e^{it} = z_0(\cos t, \sin t)$$

which is just (4.2) with $z_0 = 1$.

2. Suppose we are given in space a force field directed toward the z axis with magnitude the distance from the z axis (Figure 4.3).

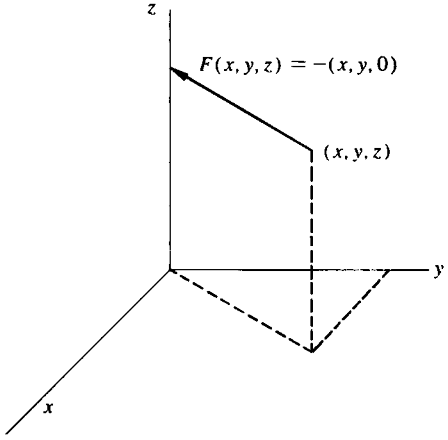


Figure 4.3

Here again the force field is independent of time and is given by

$$\mathbf{F}(x, y, z, t) = -(x, y, 0)$$

If a particle is at $(1, 0, 0)$ with an initial velocity of $(0, 1, a)$, what is its path of motion? We must solve this differential equation for three unknown functions $\mathbf{f}(t) = (x(t), y(t), z(t))$

$$\mathbf{f}''(t) = (x''(t), y''(t), z''(t)) = (x(t), y(t), 0)$$
$$x(0) = 1, y(0) = 0, z(0) = 0$$
$$x'(0) = 0, y'(0) = 1, z'(0) = a$$

The solution is easily found to be

$$\mathbf{f}(t) = (\cos t, \sin t, at)$$

Thus, if $a = 0$, the path of motion is a circle in the plane $z = 0$. If a is positive, the path of motion is an upward spiral lying over the unit circle, of slope a, and if $a < 0$, the path followed is a downward spiral (Figure 4.4).

3. *Time-independent fields.* If we are given a time-independent force field on a domain in R^2, or R^3, and we graph sufficiently many values of the field, it seems to be a broken line picture of a family of curves. In fact, there is a family of curves which fits the picture in this sense: there is a curve through each point $\mathbf{x}$ which is tangent to the vector $\mathbf{F}(\mathbf{x})$ at that point. These curves are called the **lines of force** of the field and are found by solving the differential equation

$$\mathbf{f}'(t) = \mathbf{F}(\mathbf{f}(t))$$
$$\mathbf{f}(0) = \mathbf{x}_0$$

The solution of this differential equation describes the line of force passing through the point $\mathbf{x}_0$.

Fluid Flows

The general notion of field of vectors arises in many other ways besides as force fields. Such an example which gives rise to a field is that of a fluid in motion in a certain domain in R^3. There are various ways of describing that flow. First of all, we may idealize, by assuming that at the time $t = 0$,

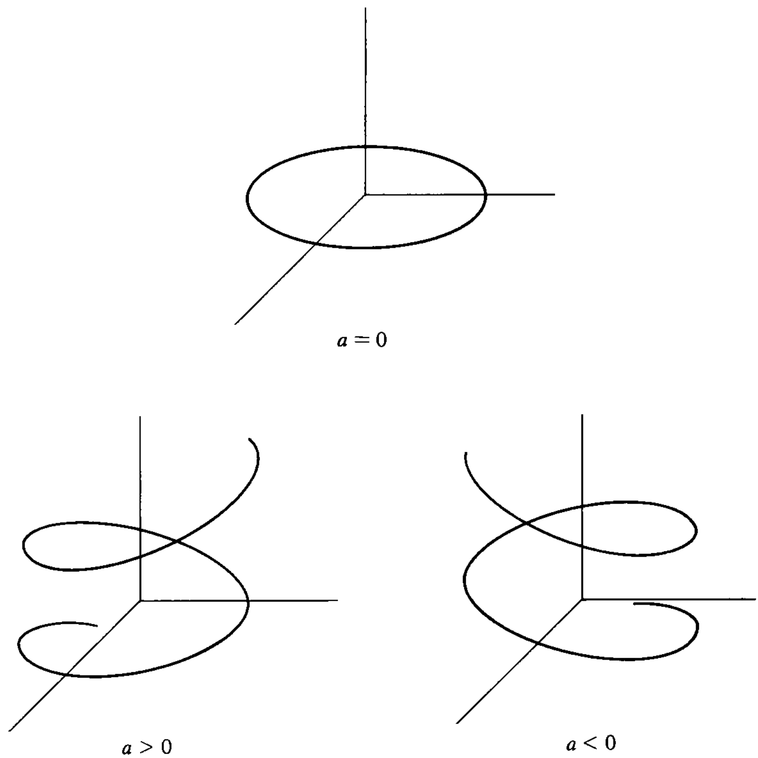


Figure 4.4

there is a particle at each point $\mathbf{x}_0$ in R^3. Then we can describe the flow by describing the motion of each particle. The particle which is at $\mathbf{x}_0$ at time $t = 0$ follows a certain path which is given by a function $\mathbf{f}(\mathbf{x}_0, t)$. The **equations of motion** are thus

$$\mathbf{x} = \mathbf{f}(\mathbf{x}_0, t)$$

Precisely, the position $\mathbf{x}$ at time t of the particle originally at $\mathbf{x}_0$ is $\mathbf{f}(\mathbf{x}_0, t)$. We assume that particles are neither created nor destroyed; this amounts to asking that, for each t the function $\mathbf{x}_0 \to \mathbf{f}(\mathbf{x}_0, t)$ is one-to-one and onto, and thus can be inverted. So we can also write

$$\mathbf{x}_0 = \phi(\mathbf{x}, t)$$

for some function ϕ. Precisely, the original position of the particle at $\mathbf{x}$ at time $-t$ was $\phi(\mathbf{x}, t)$.

4. Suppose a gas is rising at constant speed, and spiraling around the vertical axis. Thus the motion of particle is a helix as described in Example 2. We do best to express this motion in cylindrical coordinates: Let z be the (complex) coordinates in the plane ($z = re^{i\theta}$) and w the height off the plane. Thus the path of motion described by the gas is

$$(z, w) = (z_0 e^{it}, at + w_0) \tag{4.3}$$

Thus the particle originally at (z_0, w_0) will be at $z_0 e^{it}$, $w_0 + at$ at time t. We can certainly invert these equations:

$$(z_0, w_0) = (ze^{-it}, w - at) \tag{4.4}$$

Now, another way to describe a fluid flow is by its velocity. Let $\mathbf{v}(\mathbf{x}, t)$ be the velocity of the particle which is at position $\mathbf{x}$ at time t. The field $\mathbf{v}$ is called the **velocity field of the flow**. We can find the equations of motion from the velocity field by solving the appropriate differential equation. For the function $\mathbf{f}(\mathbf{x}_0, t)$ describes the motion of the particle originally at $\mathbf{x}_0$. The velocity of this particle at time t is $\mathbf{f}'(\mathbf{x}_0, t)$ and its position is $\mathbf{f}(\mathbf{x}_0, t)$. Thus we must have

$$\mathbf{f}'(\mathbf{x}_0, t) = \mathbf{v}(\mathbf{f}(\mathbf{x}_0, t), t)$$
$$\mathbf{f}(\mathbf{x}_0, 0) = \mathbf{x}_0$$

This equation can be solved uniquely.

5. Let us find the velocity field of the gas flow in Example 4. The flow equations are (4.3). The velocity of the particle originally at $\mathbf{x}_0$ is

$$(z', w') = (iz_0 e^{it}, a)$$

To find the velocity field we must write this as a function of position at time t, rather than original position. We can do this by means of the inversion (4.4), obtaining as velocity field

$$\mathbf{v}(z, w) = (iz, a)$$

6. Suppose a fluid on the plane is spiraling in toward the origin (Figure 4.5) according to this equation of flow

$$z(t) = \frac{z_0}{t} e^{it}$$

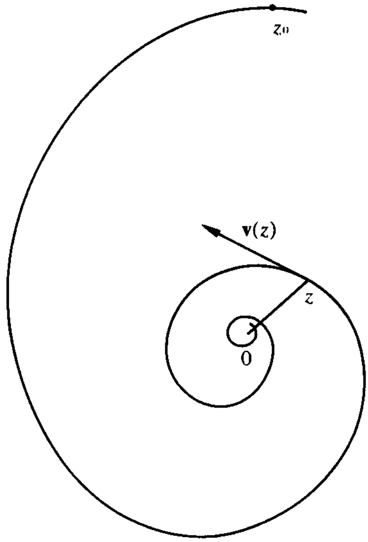


Figure 4.5

Here the particle at time $t = 1$ moves toward the origin so that its argument is proportional to time elapsed, and its distance from the origin is inversely proportional to time. Then

$$z'(t) = \frac{iz_0 e^{it}}{t} - \frac{z_0 e^{it}}{t^2} = \left(i - \frac{1}{t}\right)z(t)$$

Thus the velocity field is

$$v(z, t) = \left(i - \frac{1}{t}\right)z$$

The angular velocity is thus constant whereas the radial velocity decreases as time goes on.

7. Suppose now a fluid spiraled in toward the origin so that its velocity field was time independent, for example,

$$v(z) = (i - 1)z$$

The equations of motion are the solutions of

$$f'(t) = (i - 1)f(t)$$
$$f(0) = z_0$$

This gives

$$f(z) = e^{(-1+i)t} = e^{-t}e^{it}$$

In this case the distance from the origin decreases exponentially with time (Figure 4.6).

We shall make a study of the geometry of paths of motion of single particles and fluid flows, or families of motions, in this chapter. This study is a continuation of analytic geometry, and begins the subject of differential geometry.

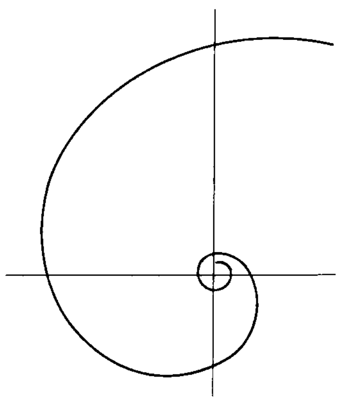


Figure 4.6

4.1 Parametrization of Curves

A curve in R^n is a one-dimensional subset Γ of R^n. This means that the set Γ can be put into one-to-one correspondence with a line, in a smooth way. We make this notion a little more precise.

Definition 1. The image in R^n of an interval under a continuously differentiable one-to-one function with a nowhere vanishing derivative is called a C^1 **curve**. If the function is k-times continuously differentiable we shall call this curve a C^k **curve**. The particular function is called a **parametrization** of the curve.

Examples

8. The unit circle in R^2 is a curve. It has this parametrization:

$$\Gamma\colon z(t) = (\cos t, \sin t) \qquad t \in R \tag{4.5}$$

Since $z'(t) = (-\sin t, \cos t)$ is never zero (the sine and cosine are never simultaneously zero), this is a good parametrization.

We could also parametrize the unit circle in this way:

$$z(t) = (t, (1 - t^2)^{1/2}) \tag{4.6}$$

but this parametrization fails at $t = \pm 1$, since the function $(1 - t^2)^{1/2}$ is not differentiable there. Notice that (4.6) does not parametrize the whole circle, but only the upper semicircle. Both of these failings can be alleviated by introducing parametrizations which cover the other parts of the circle. That is,

$$z(t) = ((1 - t^2)^{1/2}, t)$$

will parametrize the circle in the right half-plane,

$$z(t) = (t, -(1 - t^2)^{1/2})$$

takes care of the lower semicircle, and so on.

9. It is often convenient to use complex notation to describe curves in the plane. For example, the parametrization of the circle (4.5) can be written as

$$z(t) = \cos t + i \sin t = e^{it}$$

Another curve is the spiral:

$$z(t) = e^{ct}$$

where c is some complex number. Writing $c = a + ib$, this becomes

$$z(t) = e^{at}e^{ibt}$$

or, in polar notation, $z = re^{i\theta}$

$$r(t) = e^{at} \qquad \theta(t) = bt$$

Thus the modulus of z varies exponentially with t, and the argument is linear in t (see Figures 4.7 and 4.8)

10. The curve Γ:

$$\mathbf{x}(t) = (\sin t, \cos t, t) \tag{4.7}$$

called a **right circular helix**, is pictured in Figure 4.9. Since

$$\mathbf{x}'(t) = (\cos t, -\sin t, 1)$$

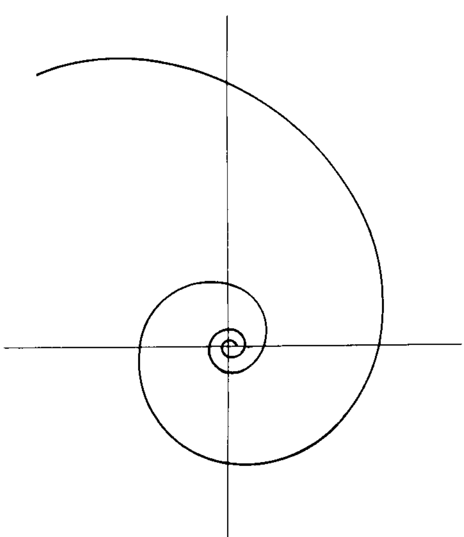
is never zero, (4.7) is a valid parametrization of the curve.

11. The intersection of two cylinders with different axes is a curve (see Figure 4.10). Suppose the cylinders are both of radius 1 and one, C_1, has as axis the y axis, and the other, C_2, has as axis the x axis. Then C_1 has the equation

$$x^2 + z^2 = 1 \tag{4.8}$$

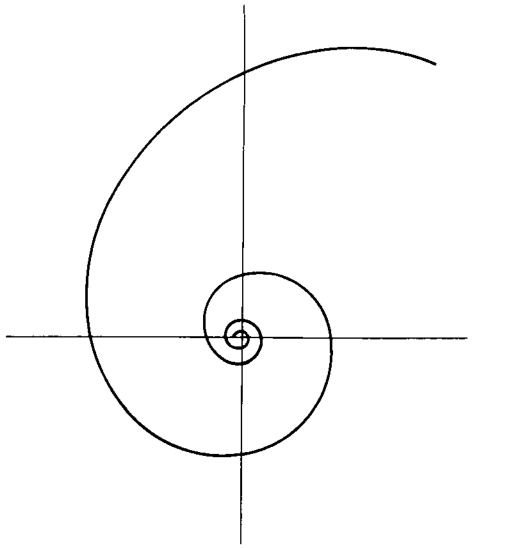
and C_2 has the equation

$$y^2 + z^2 = 1 \tag{4.9}$$



$z(t) = e^{at}$, $\operatorname{Re} a > 0$

Figure 4.7



$z(t) = e^{at}$, $\operatorname{Re} a < 0$

Figure 4.8

The intersection is, of course, the set of points where both equations hold and thus can be written $x^2 = 1 - z^2$, $y^2 = 1 - z^2$. We can thus parametrize at least part of the curve by

$$x = (1 - z^2)^{1/2} \qquad y = (1 - z^2)^{1/2} \qquad \text{or}$$
$$\mathbf{f}(t) = ((1 - t^2)^{1/2}, (1 - t^2)^{1/2}, t)$$

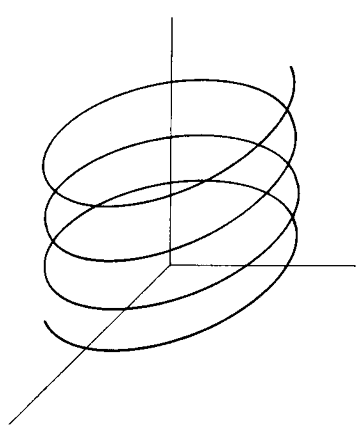


Figure 4.9

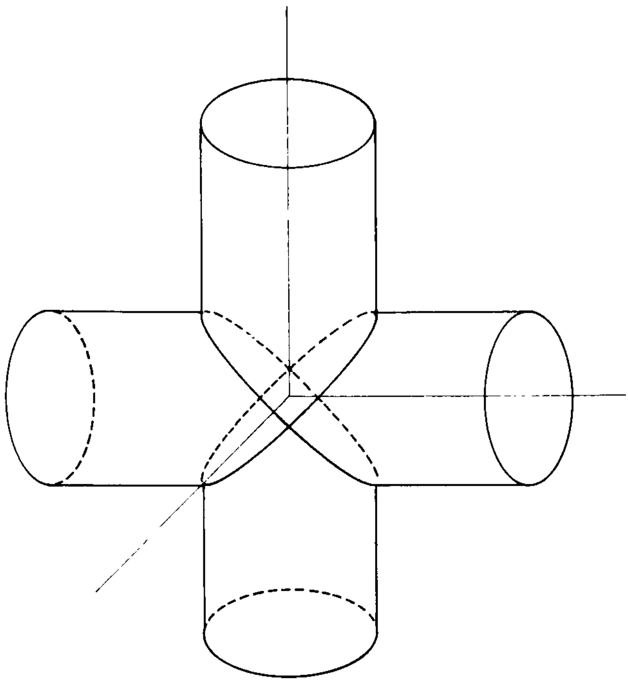


Figure 4.10

Other parts will be found by variations on this theme:

$$\mathbf{f}(t) = (-(1 - t^2)^{1/2}, (1 - t^2)^{1/2}, t)$$
$$\mathbf{f}(t) = (t, t, (1 - t^2)^{1/2})$$

and so on.

A simpler parametrization is found by the substitution $x = \cos t$. Then we have the two distinct branches of the intersection given by

$$\mathbf{f}_1(t) = (\cos, t, \cos t, \sin t)$$
$$\mathbf{f}_2(t) = (\cos, t, -\cos t, \sin t)$$

Implicitly Defined Curves

In the situation of the above example, we say that the curve is given **implicitly** by the Equations (4.8) and (4.9). More often than not, when we are given a collection of equations such as these, we can determine, just by working with them, whether or not they do implicitly define a curve. Nevertheless, the theoretical question remains: under what conditions can the set defined by a collection of equations be parametrized as a curve? We have already answered this question in R^2 in Theorem 2.14. We shall restate the conclusion as a fact about curves.

Proposition 1. *Suppose that F is a differentiable real-valued function defined in a neighborhood of (a_0, b_0) and $F(a_0, b_0) = 0$ but $dF(a_0, b_0) \neq 0$. Then the set*

$$\{(x, y) \in N \colon F(x, y) = 0\} \tag{4.10}$$

is a curve in some neighborhood N of (a_0, b_0).

Proof. Since $dF(a_0, b_0) \neq 0$, then either $(\partial F/\partial x)(a_0, b_0) \neq 0$ or $(\partial F/\partial y)(a_0, b_0) \neq 0$. Suppose the latter. Then, according to Theorem 2.16, there is an $\varepsilon > 0$ and a differentiable function g defined on the interval $(a_0 - \varepsilon, a_0 + \varepsilon)$ such that $F(x, y) = 0$ if and only if $y = g(x)$. In particular, $g(a_0) = b_0$. Let $f\colon (a_0 - \varepsilon, a_0 + \varepsilon) \to R^2$ be defined by $f(t) = (t, g(t))$. Then f parametrizes the set (4.10) near (a_0, b_0), and clearly $f'(t) = (1, g'(t)) \neq 0$. If instead $(\partial F/\partial x)(a_0, b_0) \neq 0$ we can give the same argument merely by changing the roles of x and y.

In higher dimensions the situation is a little more complicated. We shall describe it in R^3. *If F, G are two differentiable functions defined in a neighborhood of a point $\mathbf{p}_0$, and $\nabla F(\mathbf{p}_0)$, $\nabla G(\mathbf{p}_0)$ are independent, then the set*

$$\{\mathbf{p}\colon F(\mathbf{p}) - F(\mathbf{p}_0) = 0, G(\mathbf{p}) - G(\mathbf{p}_0) = 0\} \tag{4.11}$$

is a curve through $\mathbf{p}_0$.

The verification of this fact is basically another use of the fixed point theorem, complicated by some more linear algebra. We first assume that $F(\mathbf{p}_0) = 0 = G(\mathbf{p}_0)$. Since the vectors $\nabla F(\mathbf{p}_0)$, $\nabla G(\mathbf{p}_0)$ are independent, we can change coordinates in R^3 so that $\nabla F(\mathbf{p}_0) = \mathbf{E}_2$ and $\nabla G(\mathbf{p}_0) = \mathbf{E}_3$. That is, with respect to the new coordinates (x, y, z), $\partial F/\partial x(\mathbf{p}_0) = 0$, $\partial F/\partial y(\mathbf{p}_0) = 1$, $\partial F/\partial z(\mathbf{p}_0) = 0$ and $\partial G/\partial x(\mathbf{p}_0) = 0$, $\partial G/\partial y(\mathbf{p}_0) = 0$, $\partial G/\partial z(\mathbf{p}_0) = 1$. Now let $\mathbf{p}_0 = (x_0, y_0, z_0)$; for x near x_0 we want to show that there are uniquely determined y, z such that

$$F(x, y, z) = 0 \qquad G(x, y, z) = 0$$

Following Newton's method, we ask to find the fixed point in the y, z plane of the transformation

$$T(y, z) = \left(y + \frac{\partial F}{\partial y}(\mathbf{p}_0)^{-1}F(x, y, z), z + \frac{\partial G}{\partial z}(\mathbf{p}_0)^{-1}G(x, y, z)\right)$$

Our conditions $\nabla F(\mathbf{p}_0) = \mathbf{E}_2$, $\nabla G(\mathbf{p}_0) = \mathbf{E}_3$ will guarantee that in some neighborhood of $\mathbf{p}_0$, T is a contraction. Thus there are unique $y = g(x)$, $z = h(x)$ such that

$$T(g(x), h(x)) = (g(x), h(x))$$

or

$$F(x, g(x), h(x)) = 0 = G(x, g(x), h(x))$$

Thus the function $f(t) = (t, g(t), h(t))$ parametrizes the set (4.11) as a curve.

Examples

12. At what points in the plane is the set $e^{x+y} = y$ a curve? Let $F(x, y) = e^{x+y} - y$. Then $\nabla F(x, y) = (e^{x+y}, e^{x+y} - 1)$. Since $\partial F/\partial x$ is never zero, this is everywhere a curve and the equation $e^{x+y} - y = 0$ determines x as a function of y implicitly. $\partial F/\partial y$ is zero when $x + y = 0$. The only point on the curve where $e^{x+y} = y$ and $x + y = 0$ is $(-1, 1)$, so at that point we cannot expect to find y as a function of x.

Notice, that even though we cannot explicitly determine the function $x = f(y)$ given implicitly by $e^{x+y} = y$, we can find its derivative. For

$$\exp(f(y) + y) - y = 0$$

so upon differentiating we have

$$\exp(f(y) + y)(f'(y) + 1) - 1 = 0$$

or

$$f'(y) = \exp[-(f(y) + y)] - 1$$

13.

$$F(x, y) = x \sin xy - \cos y \tag{4.12}$$

$\nabla F(x, y) = (\sin xy + xy \cos xy, x^2 \cos xy + \sin y)$
If $x > 1$, $\partial F/\partial y(x, y) \neq 0$, so (4.12) defines y implicitly as a function of x. Differentiating (4.12) with respect to x we find

$$\sin xy + x \cos(xy)(y + xy') + y' \sin y = 0$$

or

$$y' = -\frac{\sin xy + xy \cos xy}{\sin y + x^2 \cos xy}$$

14. $F(x, y, z) = x^3y + y^2$, $G(x, y, z) = xyz + e^z$.

$$\nabla F = (3x^2y, x^3 + 2y, 0) \qquad \nabla G = (yz, xz, xy + e^z)$$

∇F and ∇G are dependent when

$$\frac{3x^2y}{yz} = \frac{x^3 + 2y}{xz} = \frac{0}{xy + e^z}$$

These equations become

$$xy + e^z = 0 \quad \text{and} \quad y = x^3$$

or

$$3x^2y = 0 = x^3 + 2y$$

The first pair has no solutions, and the second pair amounts to $x = 0$ and $y = 0$. But the set $F(x, y, z) = 0$, $G(x, y, z) = 0$ never intersects this plane, so everywhere on that set F and G are independent. Thus

$$\{(x, y, z): F(x, y, z) = G(x, y, z) = 0\}$$

is a curve in R^3.

Comparison of Parametrizations

Now, we have seen that a given curve admits many parametrizations, and it would be to our advantage to be able to single out a best possible one. In the study of the motion of particles there is a distinguished parameter, that of time. But as far as the geometric study is concerned we can take any parametrization we care to, the only criterion being that of convenience.

Geometrically, a most convenient parameter, or measure, along the curve is that of length as measured from a fixed point.

Before considering the particular parametrization by arc length, let us first see how to compare two different parametrizations. Suppose Γ is a curve, parametrized by

$$x = f(t) \qquad a \leq t \leq b$$

If σ is a continuously differentiable function with nonzero derivative defined on the interval $[\alpha, \beta]$ and taking values on the interval $[a, b]$, then the composed function $f \circ \sigma$ also parametrizes Γ. That is, we can write Γ as the image of

$$x = g(\tau) = f(\sigma(\tau)) \qquad \alpha \leq \tau \leq \beta$$

If τ increases as t does, then these two parametrizations determine the same sense of direction along the curve Γ. This sense of direction is called **orientation.** We know from calculus that the necessary and sufficient condition for t, τ to increase simultaneously along the curve is that $\sigma' > 0$ on the interval $[\alpha, \beta]$. We shall say that τ is an **orientation-preserving parameter** if this condition is satisfied, and otherwise τ is **orientation reversing.**

On the other hand, if we started out with two different parametrizations of a curve

$$\Gamma: x = f(t) \quad \text{or} \quad x = g(\tau) \tag{4.13}$$

then there must exist a function σ relating the two parameters. For each point of Γ corresponds to precisely one value of t and precisely one value of τ. The correspondence

$$\tau \to g(\tau) = f(t) \to t$$

defines the function σ. We shall verify below that σ is a differentiable function of τ and we have $g(\tau) = f(\sigma(\tau))$.

Notice that, given the two parametrizations, so that $t = \sigma(\tau)$, we have by the chain rule

$$g'(\tau) = f'(\sigma(\tau)) \cdot \sigma'(\tau) \tag{4.14}$$

Thus the vectors $g'(\tau)$ and $f'(t)$ are collinear when t, τ are the same points, and point in the same direction when $\sigma' > 0$, that is, when g, f induce the same orientation along Γ.

Definition 2. Let Γ be a curve parametrized by $x = f(t)$, $a \leq t \leq b$. The **unit tangent vector to Γ at** $f(t)$ is the vector

$$T(t) = \frac{f'(t)}{|f'(t)|}$$

By the above remarks we see that the unit tangent vector is the same no matter what parametrization we choose so long as it induces the same orientation. For if we have the two parametrizations (4.13), then by (4.14) (since $\sigma' > 0$)

$$\frac{f'(\tau)}{|g(\tau)|} = \frac{f'(\sigma(\tau))\sigma'(\tau)}{|f'(\sigma(\tau)) \cdot \sigma'(\tau)|} = \frac{f'(t)}{|f'(t)|}$$

when t, τ determine the same point of Γ.

Examples

15. Consider the unit circle, given parametrically by

$$z = e^{it}$$

Then $z' = ie^{it}$, which is a unit vector, so $T = ie^{it}$.
Notice: we have $T = iz$, so that the tangent vector is orthogonal to the position vector.

More generally, consider the spiral $z = e^{at}$, where a is a complex number. Then $z' = ae^{at}$, so the tangent vector is $\exp i(\text{Im } a + \arg a)t$. Notice that the angle between the tangent vector and the position vector is

$$\arg T - \arg z = \arg a$$

Thus T, z always make the same angle.

16. For the curve in space given by

$$x = (t, t^2, t^3)$$

we have

$$\frac{dx}{dt} = (1, 2t, 3t^2)$$

so

$$T(t) = \frac{1}{(1 + 8t^2 + 9t^4)^{1/2}}(1, 2t, 3t^2)$$

Now, here is the verification of the fact that two parametrizations are related by a continuously differentiable function.

Proposition 2. *Let Γ be a curve, and $f: [a, b] \to \Gamma$, $g: [\alpha, \beta] \to \Gamma$ two parametrizations of Γ. Then there is a continuously differentiable function σ mapping $[a, b]$ one-to-one onto $[\alpha, \beta]$ such that $g(\tau) = f(\sigma(\tau))$ for all $\tau \in [\alpha, \beta]$, and $f(t) = g(\sigma^{-1}(t))$, for all $t \in [a, b]$.*

Proof. Let $\tau \in [\alpha, \beta]$. Since f maps $[a, b]$ one-to-one onto Γ there is precisely one $t \in [a, b]$ such that $f(t) = g(\tau)$. Define $\sigma(\tau) = t$. Then σ is a well-defined function from $[\alpha, \beta]$ to $[a, b]$. σ is one-to-one. Suppose $\sigma(\tau_1) = \sigma(\tau_2)$. Then

$$g(\tau_1) = f(\sigma(\tau_1)) = f(\sigma(\tau_2)) = g(\tau_2)$$

Since g is one-to-one we must have $\tau_1 = \tau_2$.

σ maps $[\alpha, \beta]$ onto $[a, b]$. For if $t \in [a, b]$ there is a point $\tau \in [\alpha, \beta]$ such that $f(t) = g(\tau)$. Clearly, then $t = \sigma(\tau)$.

We now have only to verify that σ is a continuously differentiable function. Let $\tau_0 \in [\alpha, \beta]$ and $t_0 = \sigma(\tau_0)$. Now f is a differentiable function at t_0 and $f'(t_0) \neq 0$. Let $f = (f_1, \ldots, f_n)$ in coordinates. There is a j such that $f_j'(t_0) \neq 0$. Then f_j is a real-valued continuously differentiable function of a real variable and since $f'_j(t_0) \neq 0$, it is invertible. That is, there is a function h defined on the range of f_j near t_0 such that $h(f_j(t)) = t$ for t near t_0. h is also continuously differentiable. Now, since $f(\sigma(\tau)) = g(\tau)$, we have $f_j(\sigma(\tau)) = g_j(\tau)$, so

$$\sigma(\tau) = (f_j(\sigma(\tau)) = (h \circ g_j)(\tau)$$

Since h and f_j are continuously differentiable so is σ. The proposition is proven.

Without the requirement that the derivative of the parametrization is nonzero we would in general not have such a good relationship between different parametrizations. Notice that by the same argument the inverse mapping σ^{-1} to σ is also continuously differentiable. Since $\sigma^{-1} \circ \sigma(t) = t$, for all $t \in (a, b)$, we must have, by the chain rule,

$$(\sigma^{-1})'(\sigma(t)) \cdot \sigma'(t) = 1$$

so $\sigma'(t)$ is also never zero. If it is always positive, σ is an increasing function of t; if always negative σ is a decreasing function of t. Notice that if f, g are two parametrizations of a curve and they do reverse orientation, then they will become compatible simply by negating one of the parameters. Thus if f is not compatible with g, then $\tilde{f}$: $[-b, -a] \to C$ defined by $\tilde{f}(t) = f(-t)$ certainly is.

If f: $[a, b] \to \Gamma$ is a parametrization of a curve we shall call $f(a)$ the **left end point of** Γ and $f(b)$ the **right end point.**

The Tangent Line

Now, let Γ be a curve in R^n, and x_0 a point on Γ. The **tangent line** to Γ at x_0 is the straight line through x_0 which best approximates the curve. We shall show that this is the line through the tangent vector and is given by this equation

$$x = x_0 + tT(x_0) \qquad t \in R$$

The tangent line at x_0 can be computed as the limiting position of lines through x_0 and nearby points x_1 on Γ, as $x_1 \to x_0$ (Figure 4.11). Let $L(x_1)$ be that line. Then $L(x_1)$ is the set of all vectors originating at x_0 and parallel to $x_1 - x_0$. Let f give a parametrization of Γ so that $x_0 = f(t_0)$, $x_1 = f(t_1)$. Now $L(x_1)$ is the set of points x such that $x - x_0$ is parallel to

$$x_1 - x_0 = f(t_1) - f(t_0)$$

But that is the same as the set of points x such that $x - x_0$ is parallel to

$$\frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

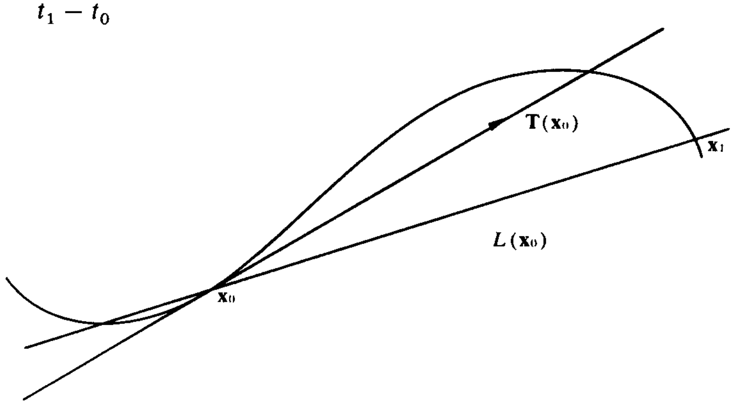


Figure 4.11

Now $x_1 \to x_0$ is the same as $t_1 \to t_0$ and the limit of the difference quotient as $t_1 \to t_0$ is $f'(t_0)$. Thus $L(x_1)$ tends to the line through x_0 and parallel to $f'(t_0)$, as desired.

Examples

17. Consider the helix (Figure 4.9), given by the parametrization

$$\mathbf{f}(t) = (a \cos t, a \sin t, bt)$$

Then

$$\mathbf{f}'(t) = (-a \sin t, a \cos t, b)$$

$\mathbf{f}$ is a positive parametrization if we take for the unit tangent

$$\mathbf{T}(t) = \frac{1}{(a^2 + b^2)^{1/2}}(-a \sin t, a \cos t, b) \tag{4.15}$$

(see Figure 4.12).

18. A damped helix (Figure 4.13) parametrized by

$$\mathbf{f}(t) = (e^t \cos t, e^t \sin t, bt)$$

Thus

$$\mathbf{f}'(t) = (e^t(\cos t - \sin t), e^t(\sin t + \cos t), b)$$

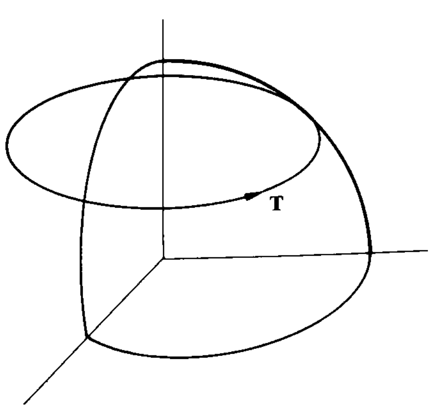


Figure 4.12

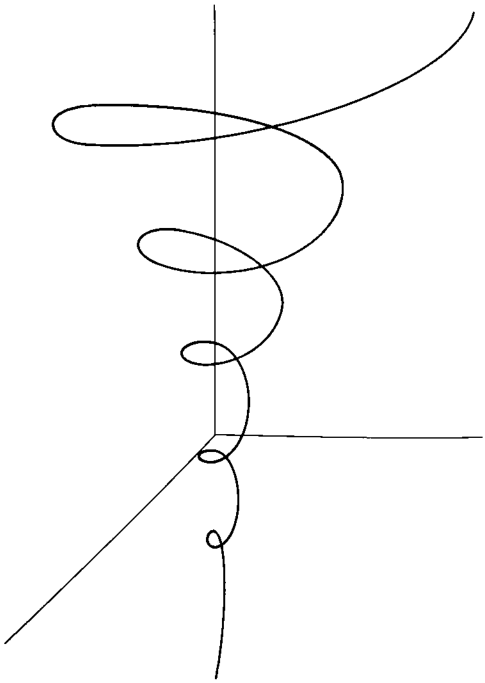


Figure 4.13

so we can take as the tangent vector

$$\mathbf{T}(t) = \frac{1}{(2e^{2t} + b^2)^{1/2}} (e^t (\cos t - \sin t), e^t(\sin t + \cos t), b) \tag{4.16}$$

Notice that the curve on the unit sphere swept out by the tangent is the same for both helices (Figure 4.14), and that the functions (4.15) and (4.16) give two different parametrizations of this curve. If we consider the parameter as t, then the "moving point" described by (4.15) has no tangential acceleration, whereas in (4.16) it is accelerating exponentially.

19. A different helix is this one (Figure 4.15):

$$\mathbf{f}(t) = (\cos t, \sin t, e^t)$$

Here we take as tangent vector

$$\mathbf{T}(t) = \frac{1}{(1 + e^{2t})^{1/2}} (-\sin t, \cos t, e^t)$$

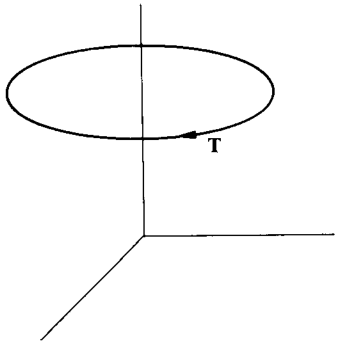


Figure 4.14

(Figure 4.16). This again is a helix on the unit sphere which tends to the equator as $t \to -\infty$ and winds rapidly around the north pole as $t \to +\infty$. (Notice that

$$z(\mathbf{T}(t)) = \frac{1}{(1+e^{-2t})^{1/2}} \quad \begin{array}{l} \to 1 \text{ as } t \to \infty \\ \to 0 \text{ as } t \to -\infty) \end{array}$$

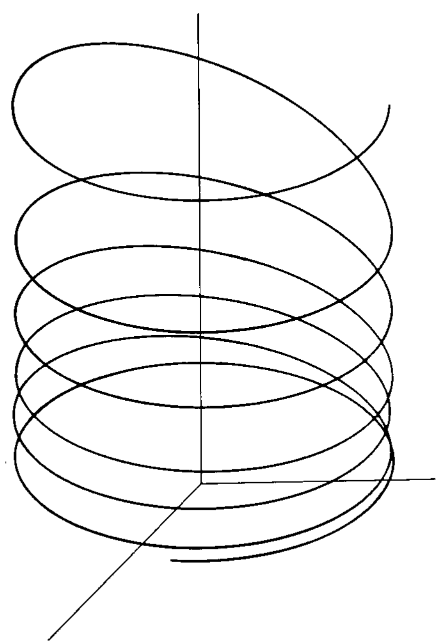


Figure 4.15

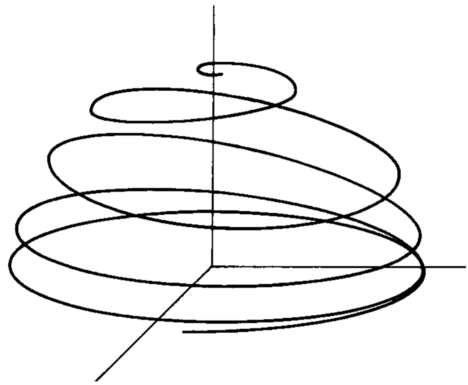


Figure 4.16

20. The intersection of a sphere and a cylinder (Figure 4.17)

$$x^2 + y^2 + z^2 = 1$$

$$(x - \tfrac{1}{2})^2 + y^2 = \tfrac{1}{4}$$

In order to avoid the cross at $(1, 0, 0)$ we shall restrict attention to the part of the curve lying above the xy plane. Let us first parametrize this curve. We shall use as parameter the angle θ as shown in the figure. Then

$$x(\theta) = \tfrac{1}{2} + \tfrac{1}{2}\cos\theta \qquad y(\theta) = \tfrac{1}{2}\sin\theta$$

and $z(\theta)$ is the point on the unit sphere lying above $(x(\theta), y(\theta), 0)$, thus $z(\theta)$ is the positive square root of $1 - (x(\theta))^2 - (y(\theta))^2$, which is

$$\left(\frac{1 - \cos\theta}{2}\right)^{1/2} = \sin\frac{\theta}{2}$$

Thus, we can parametrize this curve with the function

$$\mathbf{f}(\theta) = \left(\frac{1}{2} + \frac{1}{2}\cos\theta, \frac{1}{2}\sin\theta, \sin\frac{\theta}{2}\right)$$

Then

$$\mathbf{f}'(\theta) = \frac{1}{2}\left(-\sin\theta, \cos\theta, \cos\frac{\theta}{2}\right)$$

and we can take as tangent line

$$\mathbf{T}(\theta) = \left(\frac{2}{3 + \cos\theta}\right)^{1/2}\left(-\sin\theta, \cos\theta, \cos\frac{\theta}{2}\right) \tag{4.17}$$

Notice that this does not parametrize Γ at the point $(1, 0, 0)$, since this point corresponds to both parametric values $0, 2\pi$. In fact, Γ is not a curve at the point $(1, 0, 0)$ since it does not have a unique tangent line: the limiting position to (4.17) as $x \to (1, 0, 0)$ is either $(0, 1, 1)/\sqrt{2}$ or $(0, 1, -1)/\sqrt{2}$!

• EXERCISES

1. Find a parametrization for the curve of intersection of the ellipsoid

$$x^2 + \tfrac{1}{2}y^2 + z^2 = 1$$

with the cylinder

$$x^2 + z^2 = 1$$

2. Parametrize the intersection of the paraboliod $z = x^2 + y^2$ with the unit sphere $x^2 + y^2 + z^2 = 1$.
3. At what points is the set defined (in polar coordinates in R^2) by $r(1 + a\cos\theta) = 1$ a curve? Find a parametrization of the curve.

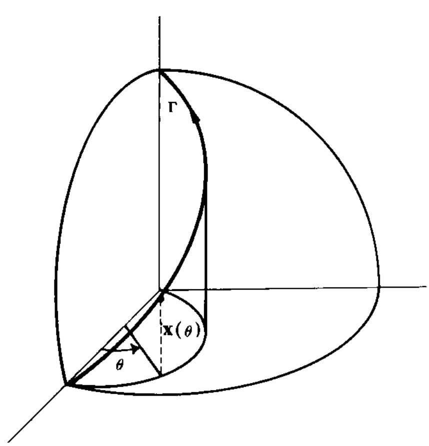


Figure 4.17

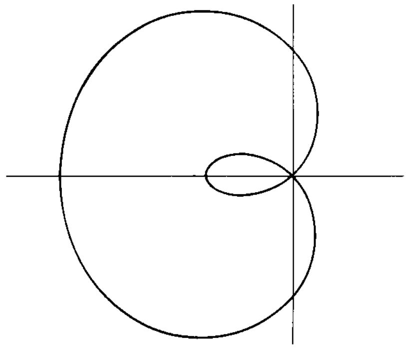


Figure 4.18

4. Consider the family of cardiods (Figure 4.18)

$$r = (1 + c)^{-1}(1 + c \cos \theta)$$

(a) Describe the behavior of this family as c ranges between 0 and $+\infty$.

(b) For $c = 1$, $c = 2$, calculate the unit tangent vector to the curve as a function of θ.

5. What is the tangent vector to the curve $r = a \cos b\theta$? Graph the curve for $b = 1, 2, 5, \sqrt{2}$.

6. Calculate the tangent lines to the following curves:

(a) $\mathbf{f}(t) = (e^{-t} \cos t, e^{-t} \sin t)$ at $(1, 0)$.

(b) $\mathbf{f}(x) = \left(x, \sin \frac{1}{x}\right)$ at $(1, 0)$.

(c) $\mathbf{f}(t) = \left(e^t, \frac{1}{t+1}, \sin t\right)$ at $(1, 1, 0)$.

(d) $x^2 + y^2 + z^2 = 4a^2$, $(x - a)^2 + y^2 = a^2$ at $(2a, 0, 0)$.

(e) $\mathbf{f}(t) = (t, \cos t, \sin t)$ at $(0, 1, 0)$.

(f) $\mathbf{f}(t) = (t^2, 1 - t^2, t)$ at $(1, 0, 1)$.

7. Find the tangent line at the origin for these curves.

(a) $e^{x+y} - y - 1 = 0$

(b) $\cos xy = y + 1$

(c) $x^2 + y^3z + \sin z = 0$ $\quad e^{xyz} - \cos xy = 0$

(d) $\exp(\sin(xy + z)) = 1$ $\quad x^2 + y^2 + z^2 = x + y + z$

• PROBLEMS

1. A snail deposits calcium at the leading edge of its shell in a direction which makes a fixed angle with the ray from the snail's center to the leading edge. Show that this hypothesis explains the spiral form of a snail's shell.

2. Graph the curve $r = (1 + \theta^2)^{-1}(2 + \theta^2)$ and compute its tangent vector.

3. Graph the curve in R^3 given in spherical coordinates by $r = e^t$, $\theta = t$, $z = e^t$. Graph the curve on the unit sphere made by the tangent vector of the given curve.

4.2 Arc Length

Definition 3. Let Γ be an oriented curve positively parametrized by $\mathbf{f}$: $[a, b] \to \Gamma$. Let $a \leq a_0 < b_0 \leq b$. Define the **length of Γ between $\mathbf{f}(a_0)$ and $\mathbf{f}(b_0)$** to be the least upper bound of all sums

$$\sum_{i=1}^{k} \|\mathbf{f}(t_i) - \mathbf{f}(t_{i-1})\| \tag{4.18}$$

over all choices of points $t_0, \ldots, t_k$ such that

$$a_0 = t_0 \leq t_1 \leq \cdots \leq t_k = b_0$$

This definition has this description. Approximate (Figure 4.19) the curve by a "broken line" joining a succession of points along Γ between a_0 and b_0. Then the sum of the lengths of the line segments is less than, and approximates the length of the curve. Now, if the points t_i and t_{i-1} are very close, then the vector $\mathbf{f}(t_i) - \mathbf{f}(t_{i-1})$ is approximately equal to $\mathbf{f}'(t_i)(t_i - t_{i-1})$. If we replace this in (4.18) we get a sum

$$\sum_{i=1}^{k} \|\mathbf{f}'(t_i)\| (t_i - t_{i-1}) \tag{4.19}$$

which is a Riemann sum approximating the integral

$$\int_{a_0}^{b_0} \|\mathbf{f}'(t)\| \, dt \tag{4.20}$$

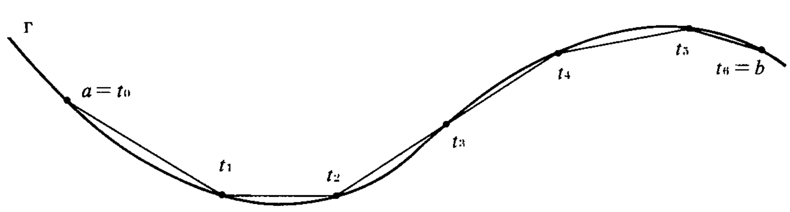


Figure 4.19

Of course, the substitutions taking us from (4.18) to (4.19) admit a small error term by term but since k may be very large, we have no hold on the error between (4.18) and (4.19). Nevertheless, we can, by being very careful, justify that substitution and deduce that the limit of the lengths of the approximating line segment curves is the integral (4.20).

Proposition 3. *Let Γ be a curve parametrized by* $\mathbf{f}: [a, b] \to \Gamma$. *The length of Γ between* $\mathbf{f}(a_0)$ *and* $\mathbf{f}(b_0)$ *is given by the integral* (4.20).

Proof. We will use the fundamental theorem of calculus to show this. Let $s(t)$ be the length of Γ between $\mathbf{f}(a_0)$ and $\mathbf{f}(t)$. We shall show that s is a differentiable function of $\mathbf{f}$, and $s'(t) = \|\mathbf{f}'(t)\|$.

Fix $t_0 \geq a_0$ and consider a $t \geq t_0$. If S_0 is any sum like (4.19) approximating the length of Γ between $\mathbf{f}(a_0)$ and $\mathbf{f}(t_0)$, then $S_0 + \|\mathbf{f}(t) - \mathbf{f}(t_0)\|$ is a sum like (4.19) for the length between $\mathbf{f}(a_0)$ and $\mathbf{f}(t)$. Thus

$$S_0 + \|\mathbf{f}(t) - \mathbf{f}(t_0)\| \leq s(t)$$

Taking the least upper bound over all such S_0, we obtain the inequality

$$s(t_0) + \|\mathbf{f}(t) - \mathbf{f}(t_0)\| \leq s(t) \tag{4.21}$$

Now, suppose S is a sum like (4.19) corresponding to a partition of the interval $[a_0, t]$. We may suppose that t_0 is one of the points in this partition. For if not, we can add it to the given partition, and get a still larger sum. Let $t_0 \leq t_1 \leq \cdots \leq t_k = t$ be the points of the partition between t_0 and t. Then

$$S = S_0 + \sum_{i=1}^{k} \|\mathbf{f}(t_i) - \mathbf{f}(t_{i-1})\|$$

where S_0 is a sum corresponding to the interval $[a_0, t_0]$. Thus

$$\begin{aligned} S &\leq s(t_0) + \sum_{i=1}^{k} \|\mathbf{f}(t_i) - \mathbf{f}(t_{i-1})\| \\ &\leq s(t_0) + \sum_{i=1}^{k} \left\| \int_{t_{i-1}}^{t_i} \mathbf{f}'(t)\, dt \right\| \\ &\leq s(t_0) + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \|\mathbf{f}'(t)\|\, dt \leq s(t_0) + \int_{t_0}^{t} \|\mathbf{f}'(t)\|\, dt \end{aligned}$$

Since this is true for all such sums S, we have

$$s(t) \leq s(t_0) + \int_{t_0}^{t} \|\mathbf{f}'(t)\|\, dt \tag{4.22}$$

From inequalities (4.21) and (4.22), we obtain

$$\frac{\|\mathbf{f}(t)-\mathbf{f}(t_0)\|}{t-t_0} \le \frac{s(t)-s(t_0)}{t-t_0} \le \frac{1}{t-t_0}\int_{t_0}^{t} \|\mathbf{f}'(t)\|\,dt \tag{4.23}$$

As $t \to t_0$, both the left and right ends converge, since $\mathbf{f}$ is differentiable, to $\|\mathbf{f}'(t_0)\|$. Thus s is differentiable at t_0, and $s'(t_0) = \|\mathbf{f}'(t_0)\|$. Since this is valid for t_0 between a_0 and b_0, we have the desired conclusion.

Now, if Γ is a curve parametrized by $\mathbf{f}$: $[a, b] \to \Gamma$, we can consider arc length as a function along Γ. Precisely, let $s(t)$ be the length of the piece of Γ from $f(a)$ to $f(t)$. Then, from the above proposition,

$$s(t) = \int_a^t \|\mathbf{f}'(t)\|\,dt$$

Since $s'(t) = \|\mathbf{f}'(t)\| > 0$, we can parametrize Γ by arc length, and it induces the same orientation as the original parametrization. Thus $\mathbf{g}(s)$, for every s is the point on Γ of distance s from a: $\mathbf{g}(s(t)) = \mathbf{f}(t)$. If L is the length of Γ from a to b, $\mathbf{g}$: $[0, L] \to \Gamma$ parametrizes Γ. Notice that

$$\begin{aligned}\mathbf{f}'(t) &= \mathbf{g}'(s(t)) \cdot s'(t)\\ &= \mathbf{g}'(s(t)) \cdot \|\mathbf{f}'(t)\|\end{aligned}$$

so that

$$\mathbf{g}'(s(t)) = \frac{\mathbf{f}'(t)}{\|\mathbf{f}'(t)\|} = \mathbf{T}(t)$$

Thus $\mathbf{g}'(s)$ is the unit tangent to Γ at $\mathbf{g}(s)$.

Examples

21. The circle $x^2 + y^2 = a^2$. Parametrize this circle by

$$x = a\cos\theta \qquad y = a\sin\theta$$

Thus

$$\begin{aligned}\mathbf{f}(\theta) &= (a\cos\theta, a\sin\theta)\\ \mathbf{f}'(\theta) &= a(-\sin\theta, \cos\theta)\\ \|\mathbf{f}'(\theta)\| &= a\end{aligned}$$

Thus arc length is given by

$$s = s(\theta) = \int_0^\theta a\, d\theta = a\theta$$

The parametrization according to arc length is thus given by substituting $s = a\theta$.

$$\mathbf{x} = \mathbf{g}(s) = \left(a \cos \frac{s}{a}, a \sin \frac{s}{a}\right)$$

The unit tangent vector is given by

$$\mathbf{T}(s) = \left(-\sin \frac{s}{a}, \cos \frac{s}{a}\right)$$

22. Consider the helix of Example 17 given by

$$\mathbf{f}(t) = (a \cos t, a \sin t, bt)$$

Then

$$\mathbf{f}'(t) = (-a \sin t, a \cos t, bt)$$
$$\|\mathbf{f}'(t)\| = (a^2 + b^2)^{1/2}$$

Thus $s = s(t) = ((a^2 + b^2)^{1/2})t$, and the arc length parametrization is

$$\mathbf{g}(s) = \left(a \cos \frac{s}{(a^2 + b^2)^{1/2}}, a \sin \frac{s}{(a^2 + b^2)^{1/2}}, \frac{b}{(a^2 + b^2)^{1/2}} s\right)$$

The tangent vector is

$$\mathbf{T}(s) = \frac{1}{(a^2 + b^2)^{1/2}}(-a \sin t, a \cos t, b)$$

23. The curve of Example 20 has the parametrization

$$\mathbf{f}(\theta) = \left(\frac{1}{2} + \frac{1}{2} \cos \theta, \frac{1}{2} \sin \theta, \sin \frac{\theta}{2}\right)$$

and we find

$$\|\mathbf{f}'(\theta)\| = \frac{1}{2\sqrt{2}}(3 + 2\cos\theta)^{\frac{1}{2}}$$

so

$$s(\theta) = \frac{1}{2\sqrt{2}}\int_0^\theta (3 + 2\cos\phi)^{1/2}\,d\phi$$

and the unit tangent vector is given as

$$\mathbf{T}(\theta) = \left(\frac{2}{3 + \cos\theta}\right)^{1/2}\left(-\sin\theta, \cos\theta, \cos\frac{\theta}{2}\right)$$

Equations of Motion

Now we shall consider in greater detail the equations of a particle in motion. Suppose a particle moves through R^n along the path given by $\mathbf{x} = \mathbf{x}(t)$. The **velocity** at time t is $\mathbf{x}'(t)$, and the **acceleration** is $\mathbf{x}''(t)$. These are vector-valued functions describing the instantaneous change in the motion (direction and magnitude) of the particle. The **speed** of the particle is the rate at which the distance covered changes, and thus is the time derivative, ds/dt, of arc length. As we have seen above, this is the magnitude of the velocity. Thus

$$\text{velocity} = \frac{d\mathbf{x}}{dt} \qquad \text{speed} = \frac{ds}{dt} = \left\|\frac{d\mathbf{x}}{dt}\right\| \tag{4.24}$$

Now, it is instinctive to decompose the acceleration vector into a component tangent to the curve, and a component orthogonal to the curve. We write

$$\text{acceleration} = \frac{d^2\mathbf{x}}{dt^2} = a_T\mathbf{T} + a_N\mathbf{N}$$

where $\mathbf{T}$ is the tangent vector and $\mathbf{N}$ is a **unit vector orthogonal to** $\mathbf{T}$ and lying in the plane spanned by the velocity and acceleration vectors. $\mathbf{N}$ is called the **principal normal to** the curve of motion, a_T is the **tangential acceleration** of the particle, and a_N is the **normal acceleration**. We now show how to compute these components of the acceleration. Differentiate the equation

$$\frac{d\mathbf{x}}{dt} = \frac{ds}{dt}\mathbf{T}$$

obtaining

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \tag{4.25}$$

Now $d\mathbf{T}/dt$ is orthogonal to $\mathbf{T}$, since $\mathbf{T}$ is a unit vector. Differentiate

$$\langle \mathbf{T}, \mathbf{T} \rangle = 1$$

We then have

$$\langle \mathbf{T}', \mathbf{T} \rangle + \langle \mathbf{T}, \mathbf{T}' \rangle = 2\,\langle \mathbf{T}, \mathbf{T}' \rangle = 0 \tag{4.26}$$

Thus we can take for the normal vector the unit vector in the direction $d\mathbf{T}/dt$

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|} = \frac{d\mathbf{T}/ds}{\|d\mathbf{T}/ds\|} \tag{4.27}$$

(Of course, the differentiation in (4.25) could have been with respect to arc length as well as time.) Let $\kappa = \|d\mathbf{T}/ds\|$. This is called the **curvature of the path of motion.** Then $d\mathbf{T}/ds = \kappa\mathbf{N}$, and (4.25) becomes

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{ds}\frac{ds}{dt}$$

$$\text{acceleration} = \frac{d^2\mathbf{x}}{dt^2} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}$$

Thus the tangential acceleration is the rate of change of the speed, and the normal acceleration is proportional to the curvature, or bending, of the curve.

$$a_T = \frac{d^2s}{dt^2} \qquad a_N = \left(\frac{ds}{dt}\right)^2\kappa \tag{4.28}$$

Examples

24. Suppose a particle moves along the parabola $y = 1 - x^2$ according to these equations

$$x = t - 1 \qquad y = 2t - t^2$$

Then

$$\mathbf{x} = (t - 1, 2t - t^2)$$

$$\frac{d\mathbf{x}}{dt} = (1, 2(1 - t))$$

$$\frac{d^2\mathbf{x}}{dt^2} = (0, -2)$$

Thus the motion of the particle is determined by a downward vertical acceleration of constant magnitude (perhaps due to gravity) (see Figure 4.20). The speed of the particle is

$$\left\| \frac{d\mathbf{x}}{dt} \right\| = (1 + 4(1 - t)^2)^{1/2} \tag{4.29}$$

Thus we see that the speed is decreasing until time $t = 1$ (the maximum height of the trajectory), and then increases. The tangent vector to the path of motion is

$$\mathbf{T} = (1 + 4(1 - t)^2)^{-1/2}(1, 2(1 - t))$$

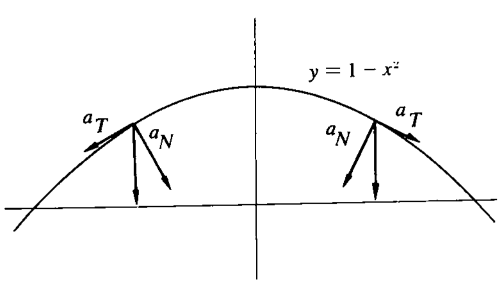


Figure 4.20

and so

$$\frac{d\mathbf{T}}{dt} = \frac{2}{[1 + 4(1 - t)^2]^{3/2}}(2(1 - t), -1) \tag{4.30}$$

The normal vector is the unit vector in this direction:

$$\mathbf{N} = (1 + 4(1 - t)^2)^{-1/2}(2(1 - t), -1)$$

Now

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\bigg/\frac{ds}{dt} = \frac{2}{(1 + 4(1 - t)^2)^2}(2(1 - t), -1)$$

$$= \frac{2}{[1 + 4(1 - t)^2]^{3/2}}\mathbf{N}$$

Thus the curvature of the path of motion is

$$\kappa = \frac{2}{(1 + 4(1 - t)^2)^{3/2}}$$

And finally

$$a_T = \frac{d^2s}{dt^2} = -\frac{4(1 - t)}{(1 + 4(1 - t^2))^{3/2}} \qquad a_N = \left(\frac{ds}{dt}\right)^2\kappa = \frac{2}{(1 + 4(1 - t)^2)^{1/2}}$$

The length of the trajectory from $x = -1$ to $x = +1$ is

$$\int_0^2 \left\|\frac{d\mathbf{x}}{dt}\right\| dt = \int_0^2 [1 + 4(1 - t)^2]^{1/2}\, dt$$

25. (Rotation) (Figure 4.21). Suppose now that a particle rotates around the unit circle according to the equations

$$x = \cos(e^t) \qquad y = \sin(e^t)$$

Then

$$\mathbf{x} = (\cos(e^t), \sin(e^t))$$

$$\frac{d\mathbf{x}}{dt} = e^t(-\sin(e^t), \cos(e^t)) \tag{4.31}$$

$$\frac{d^2\mathbf{x}}{dt^2} = e^t(-\sin(e^t), \cos(e^t)) - e^{2t}(\cos(e^t), \sin(e^t)) \tag{4.32}$$

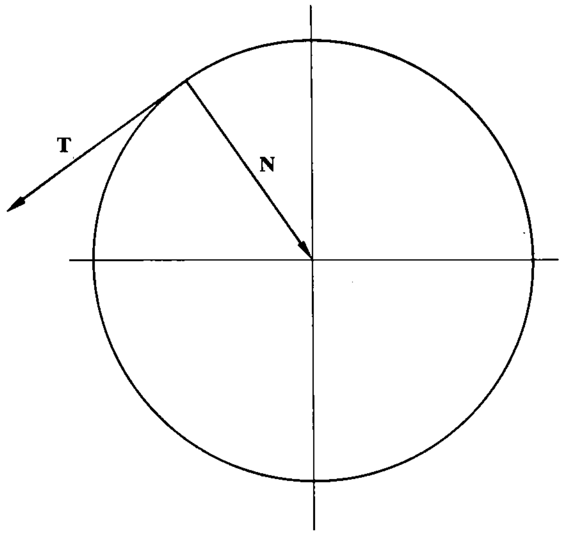


Figure 4.21

Now we already know, just from geometric considerations, what are the tangent and normal to the path of motion:

$$\mathbf{T} = (-\sin(e^t), \cos(e^t)) \qquad \mathbf{N} = -(\cos(e^t), \sin(e^t))$$

Thus (4.31) can be written as

$$\frac{d^2\mathbf{x}}{dt^2} = e^t\mathbf{T} + e^{2t}\mathbf{N}$$

Thus the normal acceleration is the square of the tangential acceleration. From (4.31) we read

$$\frac{ds}{dt} = \left\| \frac{d\mathbf{x}}{dt} \right\| = e^t$$

thus

$$s = e^t$$

and the curvature of the unit circle is 1.

Notice, that any motion on the unit circle can be written in the form

$$\mathbf{x} = (\cos(f(t)), \sin(f(t))$$

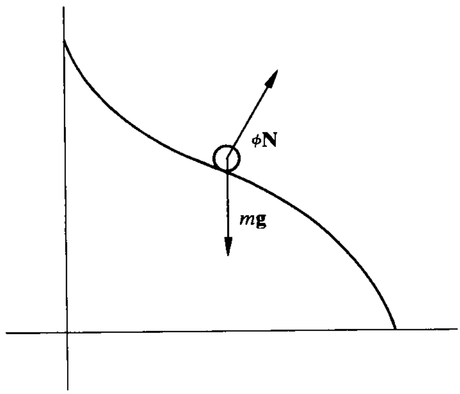


Figure 4.22

where $f(t)$ represents arc length as a function of time. Since the curvature of the unit circle is 1, we obtain for any circular motion

$$\text{acceleration} = \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2\mathbf{N}$$

The tangential acceleration is the rate of change of speed, and the normal acceleration is the square of the speed.

26. Now let us consider the motion of an object down a slide (see Figure 4.22). The slide will be represented by the curve Γ. Let $z = z(t) = x(t) + iy(t)$ be the equation of motion of the particle. The acceleration is $z''(t)$; according to Newton's laws

$$mz'' = F$$

where m is the mass of the object, and F is the sum of the forces acting on the object. One such force is the force due to gravity which is mg, where g is the gravitational field. The other force is the restraining force due to the curve. This force acts in a direction normal to the curve, and has undetermined magnitude. (That is, its magnitude is determined only by the object.) Let us call this force ϕN, where ϕ is a scalar and N is the normal to the curve. Thus we have

$$mz'' = mg + \phi N$$

Now, since we know the path of motion, we need only determine the tangential acceleration a_T. By Equation (4.28), we have

$$\frac{d^2s}{dt^2} = a_T = \langle z'', T\rangle = \langle g, T\rangle \tag{4.33}$$

where T is the tangent of the curve. If we consider the curve as parametrized by arc length: $z = f(s)$ is the equation of the curve, then the tangent vector is $f'(s)$. Then Equation (4.33) becomes

$$\frac{d^2s}{dt^2} = \langle g, f'(s)\rangle$$

and the speed can be found as the solution to this differential equation with initial conditions $s(0) = s'(0) = 0$.

For specific examples, let us first consider the curve to be a straight line (Figure 4.23) with equation

$$z(s) = i + s\xi_0$$

where $\xi_0 = a + ib$ is a unit vector in the third quadrant ($b > 0$). Then $T(t) = \xi_0$ is constant, and the force due to gravity is $-ig$. The speed

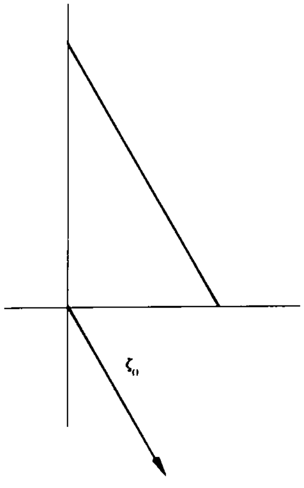


Figure 4.23

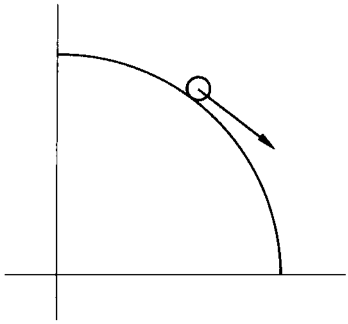


Figure 4.24

is thus found as the solution of the differential equation

$$\frac{d^2s}{dt^2} = \langle -ig, a + ib\rangle = -gb$$

$$s(0) = s'(0) = 0$$

Thus $s(t) = -(gbt^2)/2$ and the equation of motion is

$$z = z(t) = i - (\tfrac{1}{2}gbt^2)\xi_0$$

27. Suppose now the curve is a semicircle (Figure 4.24)

$$z(s) = \sin s + i \cos s$$

Then

$$T(s) = \cos s - i \sin s$$

and the speed is the solution of the differential equation

$$\frac{d^2s}{dt^2} = \langle -ig, \cos s - i \sin s\rangle = g \sin s$$

$$s(0) = s'(0) = 0$$

Rotating Plates

28. We can describe the motion of a rotating flat circular plate by referring to the angle as a function of time. Let a line through the center of the plate be chosen at time $t = 0$ and let $\theta(t)$ be the angle this line makes at time t with its original position. Then a point at

z_0 at time $t = 0$ follows the path of motion

$$z = z_0 e^{i\theta(t)}$$

Its velocity is $iz_0\theta' e^{i\theta(t)}$, so its speed is $|z_0|\theta'$. The acceleration of the point is found by differentiating further:

$$z''(t) = iz_0\theta'' e^{i\theta(t)} - z_0(\theta')^2 e^{i\theta(t)} \tag{4.34}$$

Thus the tangential acceleration is $|z_0|\theta''$ and the radial acceleration is $z_0(\theta')^2$.

If there is an object of mass m on the plate, a force $mz''(t)$ is required so that the object will follow the motion of the plate. Friction may provide this force. Notice that the central component of this force is $z_0(\theta')^2$, so even if there is no angular acceleration, friction must do its job. The further the object is from the center, or the faster the plate spins, the greater the force required. It is this principle which explains the centrifuge, which settles precipitates in solution by spinning the fluid.

29. Suppose now we have a curved circular plate spinning at constant angular velocity, and there is a ball of mass m in the plate (Figure 4.25). Assuming there is no friction, we can describe the motion of the ball in terms of the initial data.

Let us use spherical coordinates r, θ, z in R^3, so that the plate is given by the equation $z = f(r)$. In Figure 4.25 we depict a planar section of the plate. Let

$$r = r(t) \qquad \theta = \theta(t) \qquad z = z(t)$$

be the equations of motion of the ball, and let α be the angular velocity of the plate. Since there is no friction, the ball rotates as does the

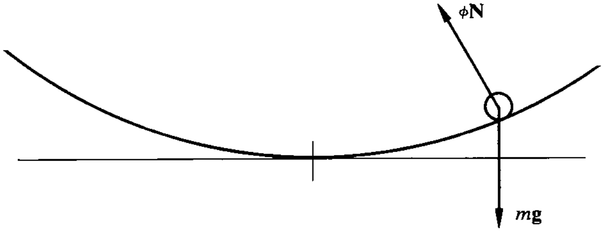


Figure 4.25

plate, then $\theta = \theta(t) = \alpha t$. Since the ball is constrained to lie on the plate we must have $z(t) = f(r(t))$, for all t. Thus we have

$$\mathbf{x}(t) = (r(t)e^{i\alpha t}, f(r(t)))$$

as the equation of motion, and we must find, using Newton's laws, the function $r(t)$. Now the acceleration is

$$\mathbf{x}'' = ((r'' - \alpha^2 r + 2i\alpha r')e^{i\alpha t}, f''(r')^2 + f'r'') \tag{4.35}$$

Letting $\mathbf{g}$ be the gravitational field, $\mathbf{g} = -(0, g)$ we have a force $m\mathbf{g}$ due to gravity. There is another force, that which restrains the motion to the profile of the plate. This acts in a direction normal to the plate and has undetermined magnitude. Let $\phi\mathrm{N}$ denote this force. There is a third force acting on the ball, due to the rotation of plate and the direction of this force is tangential to the circle on which the ball lies. We shall denote this force by $\mathbf{C}$. Then, by Newton's laws

$$\phi\mathrm{N} + \mathbf{C} + m\mathbf{g} = m\mathbf{x}'' \tag{4.36}$$

Let us equate coordinates. Now, since N is normal to the surface, it lies in the plane through the z axis and the ball (the rz plane) and is normal to the curve $z = f(r)$. Thus $\mathrm{N} = (n_1 e^{i\alpha t}, n_2)$ and

$$\frac{n_2}{n_1} = -(f'(r))^{-1}$$

(since n_2/n_1 is the slope of the line perpendicular to the curve $z = f(r)$). Since $\mathbf{C}$ is tangent to the circle on which the ball lies, $\mathbf{C} = (ce^{i\alpha t}, 0)$. The magnitude c of $\mathbf{C}$ is yet to be determined. Finally, $\mathbf{g}$ is vertical, so $\mathbf{g} = (0, g)$. Using (4.35) and substituting these values in (4.36) we have these three equations as a result:

$$\phi n_1 = m(r'' - \alpha^2 r)$$
$$c = 2\alpha r' m$$
$$-mg + \phi n_2 = m(f''(r')^2 + f'r'')$$

Thus, eliminating ϕ from the first and last equations, we find that $r = r(t)$ is a solution of the differential equation

$$(1 + f'(r)^2)r'' = \alpha^2 r - f'(r)f''(r)(r')^2 - f'(r)g \tag{4.37}$$

For what kind of a plate will it be true that the ball will not move up or down, once released no matter what its position? We must have $r' = r'' = 0$, so (4.37) becomes

$$\alpha^2 r = f'(r)g$$

Solving, we obtain $f(r) = (\alpha^2/2g)r^2$. Thus if the plate is a paraboloid of revolution, we can rotate it at a suitable angular velocity so that it will have this property.

30. Suppose we are given a field of force in space, and the initial position and velocity of a particle. Then we can find the path of motion of that particle. For example, suppose the force field is $F(x) = -x$, and the initial position and velocity of the particle are x_0, v_0. Then the path of motion is given by the solution of this differential equation:

$$f''(t) = -f(t)$$
$$f(0) = x_0 \qquad f'(0) = v_0$$

We know the solution; it is

$$f(t) = \cos t \cdot x_0 + \sin t \cdot v_0$$

Thus the path of the particle is an ellipse in the plane determined by the vectors x_0, v_0. If x_0, v_0 are orthogonal the major and minor axis have lengths $|x_0|$, $|v_0|$ (see Figure 4.26). The velocity vector is

$$f'(t) = -\sin t \cdot x_0 + \cos t \cdot v_0$$

and the speed is the length of this vector.

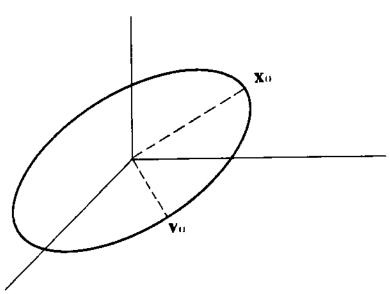


Figure 4.26

31. Suppose we have a force field in the plane which is of the same magnitude as the position vector, but orthogonal to it. Using complex variables on the plane, the force field is given by

$$F(z) = iz \quad \text{or} \quad -iz$$

Let us assume it is the former. Suppose a particle has initial position z_0 and velocity v_0. Then, the motion is found by solving

$$f''(t) = if(t) \qquad f(0) = z_0 \qquad f'(0) = v_0$$

The solution is of the form

$$f(t) = Ae^{\alpha t} + Be^{-\alpha t}$$

where $\alpha = \sqrt{i} = (1 + i)/\sqrt{2}$. We solve for A, B by substituting the initial conditions,

$$f(0) = z_0 = A + B$$
$$f'(0) = v_0 = \alpha(A - B)$$

Thus

$$f(t) = \frac{z_0 - iv_0}{2} e^{\alpha t} + \frac{z_0 + i\alpha v_0}{2} e^{-\alpha t}$$

Suppose $z_0 = 1$, $v_0 = 0$. Then

$$f(t) = \tfrac{1}{2}(e^{\alpha t} + e^{-\alpha t})$$

For large positive t, the second term is negligible, and the curve is very close to

$$z = \tfrac{1}{2}e^{\alpha t}$$

which we know is an outgoing counterclockwise spiral. For large negative t, the second term $e^{-\alpha t}$ is dominant and that gives an incoming clockwise spiral. Thus the particle comes spiraling in from outer space and then at time $t = 0$ pauses for a breath and then goes racing back from whence it came. (See Figure 4.27.)

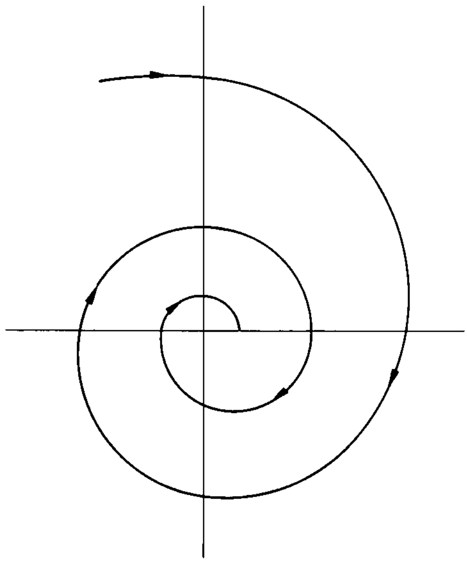


Figure 4.27

• EXERCISES

8. Find arc length as a function of the parameter for each of the following curves.
   (a) $r(1 + a \cos \theta) = 1$
   (b) $r = 1 + 2 \cos \theta$
   (c) The curves in Exercises 6(a)(b)(d)(f), and 7(a).
9. Parametrize these curves according to arc length, and find the curvature and normal.
   (a) $x^2 + y^2 = 1, x^2 + z^2 = 1.$
   (b) The curves in Exercise 8(a)(b).
   (c) The curve in Exercise 6(a)(e).
   (d) The curve of Example 22.
   (e) The curve of Example 23.
10. Find the normal and tangential accelerations for these planar motions:
   (a) $z(t) = \exp(1 - i)t$
   (b) $x(t) = t^2, y(t) = t^3$
   (c) $z(t) = (1 + 2 \cos t)e^{it}$
   (d) $z(t) = t + e^{it}$
11. Find the normal and tangential accelerations of these motions in space:
   (a) $\mathbf{x}(t) = (t, \sin t, \sin t)$
   (b) $\mathbf{x}(t) = (e^t, e^{-t}, t^2)$
   (c) $\mathbf{x}(t) = t(\sin t, \cos t, 1)$

• PROBLEMS

4. The graph of a differentiable function $y = f(x)$ is a curve in the plane. Find the curvature as a function of x.

5. The graph of a differentiable R^2-valued function $y = f(x)$, $z = g(x)$ is a curve in space. Find its curvature as a function of x.

6. A skier has to negotiate a series of hills whose profile is the curve $y = e^{-x} \cos x$ (Figure 4.28). There are three forces acting on the skier: that due to gravity, the restraining force of the hills, and a force due to friction which is proportional to his velocity. Find the differential equation describing his motion.

7. I shot an arrow into the sky at an initial velocity of 80 feet/second and at an angle of $\pi/3$ with the horizontal. The gravitational field is vertical downward with a magnitude of 32 feet/second^2. The air drags the arrow with a force of 0.05 times its velocity. Find the equation of motion, and the curvature of the curve of motion (the arrow weighs one pound).

8. In Example 26, let κ be the curvature of the slide. Show that the magnitude of the constraining force due to the slide is $f = (ds/dt)^2\kappa - \langle \mathbf{g}, \mathbf{N} \rangle$. Find the differential equations which determine $x(t)$, $y(t)$. Write out these equations when the slide is the curve $y = \cos x$.

9. Suppose we have a field of force in space given by $\mathbf{F}(x, y, z) = (-y, x, z)$. Find the path of motion of a particle which at time $t = 0$ is at $(1, 1, 1)$ with velocity $(-1, -1, 1)$.

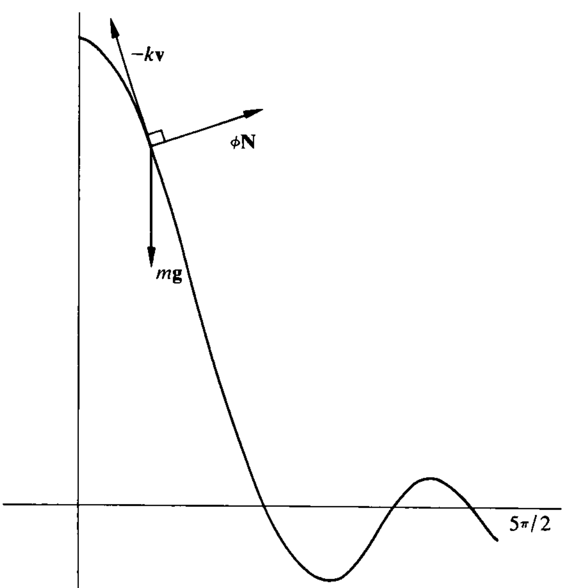


Figure 4.28

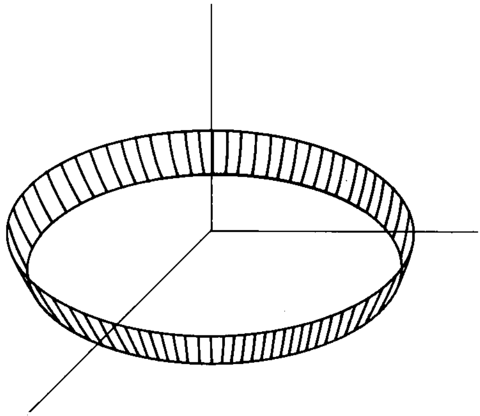


Figure 4.29

10. Suppose a race track is formed by rotating the curve $(x-1)^2 + z^2 = 1$, $-1 \leq z \leq 0$ around the z axis. (The surface is, in cylindrical coordinates, $(r-1)^2 + z^2 = 1$, Figure 4.29). A cyclist cycling around the track tends to ride up the bank as he goes faster. Explain that.

11. Water is at rest in a very large sink when a stopper is removed in the bottom center of the sink. An idealization of the ensuing motion is as follows. The water accelerates toward the hole. The forces acting on each particle of water are due to gravity and the mass of the fluid itself. The field due to the former is $-(0, 0, g)$ and the field due to the latter operates as if the particle were on an inclined plane with vertex at the hole. Find the resultant force field. Find the differential equation giving the rate of rotation around the hole.

12. We must send a ball of unit mass over a hill whose profile is the curve $y = \exp(-x^2)$ from $x = -1$ to $x = 1$. What minimum initial speed is required to ensure that the ball maneuvers this hill?

13. Suppose we are given in space a force field which is directed toward the origin and so that its component in the z direction is always 1. Find the path of motion of a particle which is at rest at time $t = 0$ at the point $(1, 1, 1)$.

4.3 Local Geometry of Curves

We have seen, from the physical problems discussed, that the higher-order derivatives of a function parametrizing a curve have some significance. In this section we will discuss the higher-order **invariants** of a curve; that is, those concepts which depend only on the geometry and not on the particular parametrization.

Let Γ be a curve in R^n. For purposes of simplicity, we shall take Γ to be parametrized by arc length by $\mathbf{x} = \mathbf{x}(s)$. If Γ is twice differentiable, the tangent vector $\mathbf{T}(s) = \mathbf{x}'(s)$ is a differentiable function. Since $\langle \mathbf{T}(s), (\mathbf{T}s)\rangle = 1$ for all s, we obtain through differentiation $2\langle \mathbf{T}(s), \mathbf{T}'(s)\rangle = 0$. Thus at any point $\mathbf{T}'$ is orthogonal to $\mathbf{T}$.

Definition 4. The **normal line** to Γ at $\mathbf{x}_0 = \mathbf{x}(s_0)$ is the line through $\mathbf{x}_0$ and parallel to the vector $\mathbf{T}'(s_0)$. The **osculating** (or **tangent**) plane to Γ at $\mathbf{x}_0$ is the plane spanned by the tangent and normal lines.

The name **osculating plane** is quite descriptive. This plane osculates in the following sense.

Proposition 4. *Let $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ be three points on the curve Γ. If they are noncollinear, they determine a plane. This plane has the osculating plane as limiting position, as $\mathbf{x}_1, \mathbf{x}_2$ tend to $\mathbf{x}_0$.*

Proof. In order to determine the limiting position of the plane through x_0, x_1, x_2 it suffices to find two independent vectors which are limits of vectors on the variable plane. The easiest way to do this is to refer to the Taylor expansion of the arc length parametrization. Suppose $f\colon (a, b) \to \Gamma$ parametrizes Γ with respect to arc length, and $f(0) = x_0$. For simplicity we may assume x_0 is the origin, 0. According to Theorem 4.1 we can write

$$f(s) = T(0)s + T'(0)s^2 + \varepsilon(s)s^2 \tag{4.38}$$

where $\lim_{s \to 0} \varepsilon(s) = 0$.

Let $x_1 = f(s_1)$, $x_2 = f(s_2)$. Since $x_0 = f(0) = 0$, the plane $\pi(s_1, s_2)$ through x_0, x_1, x_2 is the plane spanned by the vectors $f(s_1), f(s_2)$. Now, for each s_1, $s_1^{-1}f(s_1)$ is on $\pi(s_1, s_2)$. Now

$$s_1^{-1}f(s_1) = T(0) + T'(0)s_1 + \varepsilon(s)s_1^2$$

Letting $s_1 \to 0$, that says that $\lim_{s_1 \to 0} s_1^{-1}f(s_1) = T(0)$ is on the limiting plane. Now, to find another vector on the limiting plane, we take an appropriate combination of $f(s_1), f(s_2)$ so as to dispose of the $T(0)s$ term in the Taylor expansion (4.38). Thus, we consider

$$s_2 f(s_1) - s_1 f(s_2) = T'(0)(s_2 s_1^2 - s_1 s_2^2) + \varepsilon(s_1)s_2 s_1^2 - \varepsilon(s_2)s_1 s_2^2 \tag{4.39}$$

We are interested in finding some vector of this form which has a limit as s_1, s_2 tend to zero. Let us take the special case $s_1 = 2s_2 = 2s$; (4.39) becomes

$$T'(0) \cdot 2s^3 + \varepsilon(2s) \cdot 4s^3 - \varepsilon(s) \cdot 2s^3 = 2s^3(T'(0) + 2\varepsilon(2s) - \varepsilon(s))$$

Thus $T'(0) + 2\varepsilon(2s - \varepsilon(s))$ is on the plane spanned by $f(s)$ and $f(2s)$. Letting $s \to 0$, we see that $T'(0)$ is on the limiting plane. Thus the limiting plane is indeed spanned by $T(0)$ and $T'(0)$.

A few remarks are in order. If $\mathbf{T}'(0) = 0$, then the osculating plane is not defined. In particular, if the curve Γ is a straight line, then the tangent vector is constant, and there is no plane which is closest to Γ, so *a straight line has no osculating plane anywhere.* Conversely, if $\mathbf{T}$ and $\mathbf{T}'$ are always collinear along Γ, then Γ must be a straight line (Problem 14). Now, in the case where $\mathbf{T}'$ and $\mathbf{T}$ are collinear at the point in question, but not always collinear, it may happen that the plane through $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ of Proposition 3 has a limiting position as $\mathbf{x}_1, \mathbf{x}_2 \to \mathbf{x}_0$, and it may not (see Problem 14). In the former case we shall consider the normal plane as defined by the limiting position, and in the latter case, we shall say that the normal plane does not exist. Generally speaking, such cases are pathological, and we shall exclude them from further discussion.

Observe that for curves in R^2, the osculating plane is (of course) just R^2. For curves Γ in R^2, we define the **normal vector to Γ at $\mathbf{x}_0$** as that unit vector $\mathbf{N}$ on the normal line so that the sense of rotation $\mathbf{T} \to \mathbf{N}$ is counterclockwise (see Figure 4.30). Then the normal vector $\mathbf{N}$ varies continuously along the curve and the vectors $(\mathbf{T}, \mathbf{N})$ will form a "natural" orthonormal basis for R^2 along the curve (called the moving frame). In R^n for $n > 2$ there is no uniquely determined choice for a normal vector, and thus we leave the choice undetermined save that it should vary continuously along Γ.

Definition 5. Let Γ be a twice differentiable oriented curve in R^n. The **normal vector** to Γ is a choice of unit vector on the normal line which varies

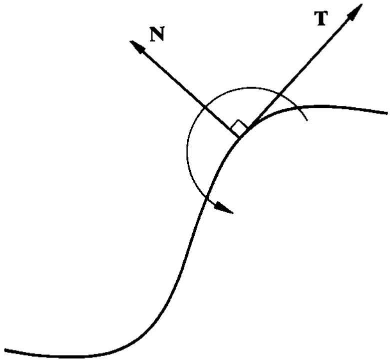


Figure 4.30

continuously along Γ. The **curvature of** Γ is the scalar function of s, $\kappa(s)$, such that $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$ along Γ.

Examples

32. The circle in R^2 (Figure 4.31)

$$\mathbf{x}(s) = \left(a \cos \frac{s}{a}, a \sin \frac{s}{a}\right)$$

$$\mathbf{T}(s) = \left(-\sin \frac{s}{a}, \cos \frac{s}{a}\right)$$

The normal is orthogonal to and counterclockwise from $\mathbf{T}$ so

$$\mathbf{N}(s) = -\left(\cos \frac{s}{a}, \sin \frac{s}{a}\right)$$

Then $\mathbf{T}'(s) = -[\cos(s/a), \sin(s/a)]/a = \mathbf{N}(s)/a$, so the curvature of the circle of radius a is a^{-1}.

33. The spiral $r = e^{\theta}$ (in polar coordinates) (Figure 4.32). The parametrization is

$$z = z(\theta) = e^{\theta}e^{i\theta} = e^{(1+i)\theta}$$

$$z'(\theta) = (1 + i)e^{(1+i)\theta}$$

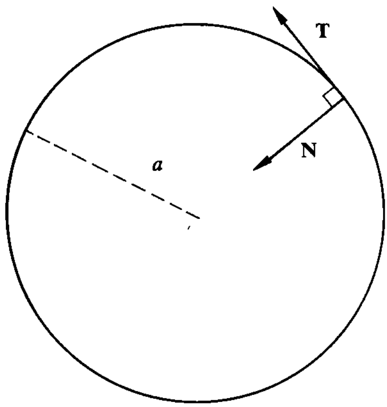


Figure 4.31

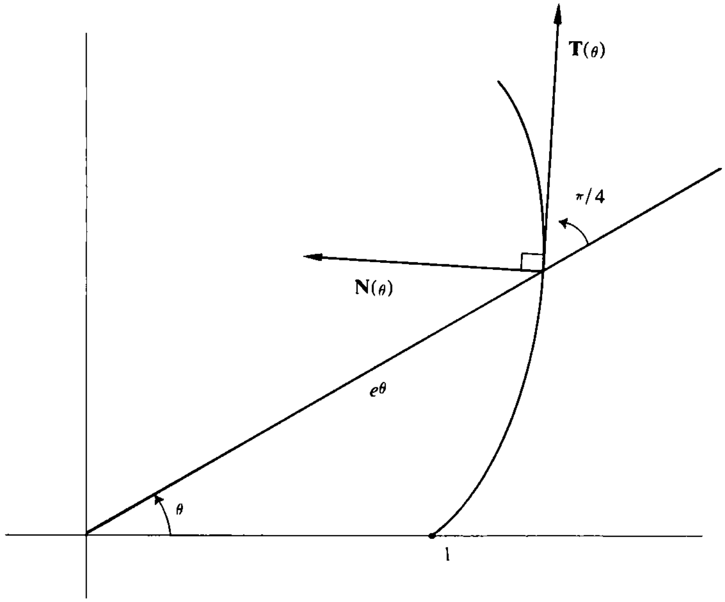


Figure 4.32

so

$$\frac{ds}{d\theta} = |z'(0)| = \frac{e^{\theta}}{\sqrt{2}}$$

Thus the tangent vector is

$$\mathbf{T}(\theta) = \frac{1+i}{\sqrt{2}} e^{i\theta} = e^{i(\theta+\pi/4)}$$

The normal is $\mathbf{N}(\theta) = e^{i(\theta+3\pi/4)}$. Now,

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{d\theta}\frac{d\theta}{ds} = ie^{i(\theta+\pi/2)}\sqrt{2}e^{-\theta} = \sqrt{2}e^{-\theta}e^{i(\theta+3\pi/4)}$$

Thus the curvature is given by $\kappa(\theta) = \sqrt{2}\,e^{-\theta}$.

Here is a proposition which gives an interpretation of curvature in the plane and sometimes makes the curvature easily computable. It says that the curvature is the rate of rotation of the moving frame with respect to arc length.

Proposition 5. *Let Γ be a given plane curve. The curvature of Γ is the rate of rotation of the tangent with respect to arc length, that is,*

$$\kappa(s) = \frac{d}{ds}(\arg \mathbf{T}(s))$$

Proof. Let $\mathbf{T}(s) = r(s)e^{i\theta(s)}$ in polar coordinates. Since $\mathbf{T}'$ is a unit vector, $r(s) = 1$. Then $\mathbf{N}(s) = e^{i(\theta(s)+\pi/2)}$, and

$$\frac{d\mathbf{T}}{ds} = \frac{d}{ds}(e^{i\theta(s)}) = i\theta' e^{i\theta} = \theta' e^{i(\theta)+\pi/2}$$

Thus $\kappa(s) = \theta'(s)$.

Examples

34. The helix

$$\mathbf{f}(t) = (a \cos t, a \sin t, bt)$$

has arc length $s = (a^2 + b^2)^{1/2}t$, and tangent vector

$$\mathbf{T}(s) = (a^2 + b^2)^{-1/2}(-a \sin(a^2 + b^2)^{-1/2}s, a \cos(a^2 + b^2)^{-1/2}s, b)$$

Thus

$$\mathbf{T}'(s) = (a^2 + b^2)^{-1}(-a \cos(a^2 + b^2)^{-1/2}s, -a \sin(a^2 + b^2)^{-1/2}s, 0)$$

Thus $\mathbf{N} = (-\cos t, \sin t, 0)$ and

$$\kappa = \frac{a}{a^2 + b^2}$$

Observe that the normal line to the helix always points toward the axis of the helix.

35. Consider the curve (Figure 4.33)

$$\mathbf{x}(t) = (\cos t, \sin t, \sin 3t)$$

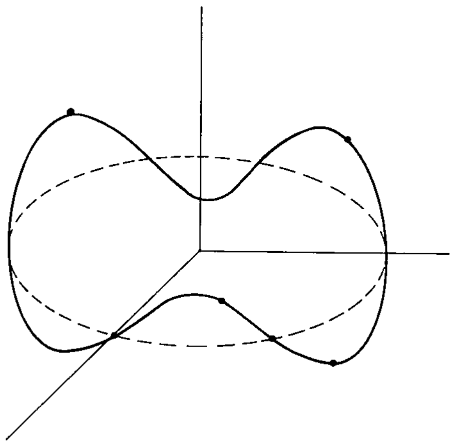


Figure 4.33

Then

$$\mathbf{x}'(t) = (-\sin t, \cos t, \cos 3t)$$

$$\frac{ds}{dt} = \|\mathbf{x}'(t)\| = (1 + 9\cos^2 3t)^{1/2}$$

$$\mathbf{T}(t) = (1 + 9\cos^2 3t)^{-1/2}(-\sin t, \cos t, \cos 3t)$$

Computing

$$\begin{aligned}\frac{d\mathbf{T}}{ds} &= \frac{d\mathbf{T}}{dt}\frac{dt}{ds} \\ &= -(1 + 9\cos^2 3t)^{-2}(10\cos t, \sin 9t, 3\cos 3t\cos 2t + \sin 3t\sin^2 t)\end{aligned}$$

and the curvature is the length of this vector.

Now, let us make one final remark about a curve in the plane. *It is completely determined, up to Euclidean motions, by its curvature.* Thus, for example, the only curve of constant curvature is a circle. This is, as we shall see, an easy consequence of Picard's existence theorem for differential equations.

Theorem 4.1. *Let $\kappa(s)$ be a continuous function of s in some interval I about the origin. There is a curve Γ whose curvature function is $\kappa(s)$. If Γ' is another curve parametrized by arc length on the interval I which has the same curvature, then a Euclidean motion will move Γ' onto Γ.*

Proof. First we shall verify the uniqueness. Let Γ be a curve with the given curvature. Let $\mathbf{x} = \mathbf{x}(s)$ be its arc length parametrization. We may apply a Euclidean motion (translation and rotation) so that $\mathbf{x}(0)$ is the origin and $\mathbf{T}(0)$ is the vector $\mathbf{E}_1$. Now we show there is only one curve with these properties. The proof depends on the observation that the normal is rigidly attached to the tangent; that is, its motion along the curve is completely determined by the tangent. In fact, writing $\mathbf{T}(s) = e^{i\theta(s)}$, we have $\mathbf{N}(s) = e^{i(\theta(s)+\pi/2)}$. Thus $\mathbf{N}' = i\theta' e^{i(\theta+\pi/2)} = \theta' e^{i(\theta+\pi)} = -\kappa\mathbf{T}$. Now the system of differential equations

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s) \qquad \mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) \tag{4.40}$$

has only one solution subject to the initial conditions $\mathbf{T}(0) = \mathbf{E}_1$, $\mathbf{N}(0) = \mathbf{E}_2$. Thus $\mathbf{T}$ is unique, so

$$\mathbf{x}(s) = \int_0^s \mathbf{T}(\sigma)\, d\sigma$$

is also uniquely determined by the given conditions. Thus there is only one Γ with the given curvature.

We now turn to the question of the existence of a plane curve with given curvature. Again, by the fundamental theorem on differential equations, there exists a solution of the system (4.40) subject to the initial conditions $\mathbf{T}(0) = \mathbf{E}_1$, $\mathbf{N}(0) = \mathbf{E}_2$. If $(\mathbf{T}(s), \mathbf{N}(s))$ is the solution, then

$$\mathbf{x} = \mathbf{x}(s) = \int_0^s \mathbf{T}(\sigma)\, d\sigma$$

defines a plane curve Γ. We must show that s is arc length along Γ. For then $\mathbf{x}''(s) = \kappa(s)\mathbf{N}(s)$, so $\kappa(s)$ *is* the curvature. To show that s is arc length we must show that $\mathbf{x}'(s) = \mathbf{T}(s)$ is a unit vector.

Now, let $\tilde{\mathbf{T}}(s) = -i\tilde{\mathbf{N}}(s)$, $\mathbf{N}(s) = -i\mathbf{T}(s)$. Then

$$\begin{aligned}
&\tilde{\mathbf{T}}(0) = -i\mathbf{E}_2 = \mathbf{E}_1 \qquad \tilde{\mathbf{N}}(0) = i\mathbf{E}_1 = \mathbf{E}_2\\
&\tilde{\mathbf{T}}'(s) = -i\mathbf{N}'(s) = -i(-\kappa(s)\mathbf{T}(s)) = \kappa(s)\tilde{\mathbf{N}}(s)\\
&\mathbf{N}'(s) = i\mathbf{T}'(s) = i\kappa(s)\mathbf{N}(s) = -\kappa(s)\mathbf{T}(s)
\end{aligned}$$

Thus $\tilde{\mathbf{T}}$, $\tilde{\mathbf{N}}$ also solve the given initial value problem. By the uniqueness, $\tilde{\mathbf{T}} = \mathbf{T}$, $\tilde{\mathbf{N}} = \mathbf{N}$. Thus $\mathbf{N} = i\mathbf{T}$, so $\mathbf{N} \perp \mathbf{T}$. It follows that

$$\frac{d}{ds}\langle \mathbf{T}(s), \mathbf{T}(s)\rangle = 2\langle \mathbf{T}(s), \mathbf{T}'(s)\rangle = 2\kappa(s)\langle \mathbf{T}(s), \mathbf{N}(s)\rangle = 0$$

so $\mathbf{T}(s)$ has constant length. Since $\mathbf{T}(0) = \mathbf{E}_1$, it is a unit vector.

• PROBLEMS

14. Show that if $\Gamma: \mathbf{x} = \mathbf{x}(s)$ is a curve in R^3 and $\mathbf{T}(s)$, $\mathbf{T}'(s)$ are everywhere collinear, then Γ is a straight line.

15. (a) Let Γ be given by

$$\mathbf{x} = (x, x^3, x^3)$$

Show that $\mathbf{T}(0)$, $\mathbf{T}'(0)$ are collinear, but Γ has an osculating plane at the origin.

(b) Let

$$g(x) = \begin{cases} x^3 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

Show that the curve Γ given by

$$\mathbf{x} = (x, -g(x), g(-x))$$

does *not* have an osculating plane at the origin.

16. Let Γ be a curve on the sphere $x^2 + y^2 + z^2 = 1$. Show that Γ is an arc of a great (i.e., diametric) circle if and only if the normal to Γ is always collinear with the position vector.

17. Show that a curve is a straight line if all its tangents are parallel.

18. Three noncollinear points in R^2 determine a circle. If, for the purposes of this exercise, we consider a straight line as a circle (of infinite radius) we may assert that any three points determine a circle. Suppose Γ is a curve in R^2 through $\mathbf{p}_0$. Following the kind of reasoning on pages 324 and 325, define the **osculating circle to** Γ **at** $\mathbf{p}_0$ and find its equation in terms of a parametrization of Γ.

19. The radius of the osculating circle is called the radius of curvature. Show that it is κ^{-1}.

20. If the osculating circle to Γ is always a straight line, deduce that Γ is a straight line.

21. Find the osculating circle at a general point of an ellipse.

22. Find the osculating circle at a general point of a parabola.

23. Show that if the osculating circle to a curve is always a circle of radius R, the curve is a circle of radius R.

24. Suppose Γ is given parametrically by arc length by $x = x(s)$, $y = y(s)$. Show that the curvature is given by

$$\kappa = x'y'' - y'x'' = [(x'')^2 + (y'')^2]^{1/2}$$

25. Show that a curve in the plane of constant curvature is a circle.

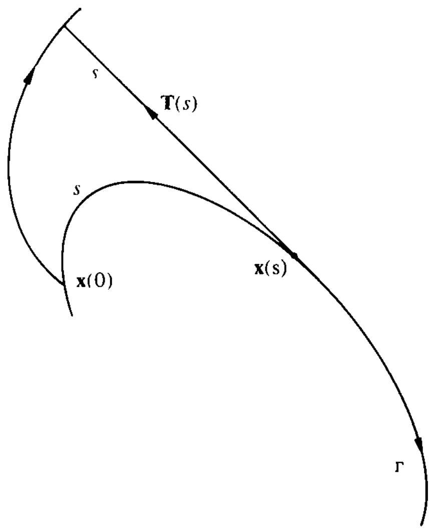


Figure 4.34

26. Suppose Γ: $\mathbf{x} = \mathbf{f}(s)$ is a curve with this property: for every t, the distance between $\mathbf{f}(s)$ and $\mathbf{f}(s + t)$ is independent of s. Show that Γ is a circle.

27. Suppose f is a nonnegative function of a real variable with the property that the area under the graph of f between 0 and x is proportional to the arc length of that graph. Find the curve.

28. Find the curve Γ with the property that at any point $\mathbf{p}$ the angle between the tangent to Γ at $\mathbf{p}$ and the tangent to the ellipse

$$E: x^2 + 2y^2 = 1$$

at the point of intersection of E with the ray through $\mathbf{p}$ is constant.

29. Let Γ: $\mathbf{x} = \mathbf{f}(s)$ be a planar curve. Suppose we have a string along Γ with one end point at $\mathbf{x}_0$. If we unwind the string tautly and without stretching, the end point will follow a curve E, called an *evolute* of Γ (Figure 4.34). If s measures arc length from $\mathbf{x} = \mathbf{f}(0)$, the curve E is parametrized by $\mathbf{x} = \mathbf{f}(s) + s\mathbf{f}'(s)$. Find the evolutes to (a) the unit circle (b) the spiral $z = e^{(1+i)t}$, (c) the parabola $y = x^2$, (d) an ellipse.

30. If we rotate a cylinder of water about its axis, the surface of the water does not remain a plane. What shape does it take and why?

4.4 Curves in Space

Suppose Γ is a curve in space. Let $\mathbf{x}_0 \in \Gamma$, and suppose $\mathbf{T}$ and $\mathbf{N}$ are the tangent and normal to Γ at $\mathbf{x}_0$. A third unit vector orthogonal to both $\mathbf{T}$ and $\mathbf{N}$ will serve to provide a natural frame within which to discuss the behavior of the curve near x_0. This vector $\mathbf{B}$, called the **binormal to the curve** is chosen so that the triple $\mathbf{T} \to \mathbf{N} \to \mathbf{B}$ forms a right-handed frame (see Figure 4.35). In this section we shall use this frame, called the **moving trihedron** along the curve, much as we used the tangent and normal to study plane curves.

The three vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ determine three planes: the *tangent* (*or osculating*) *plane* is spanned by $\mathbf{T}$ and $\mathbf{N}$, the *normal plane* is spanned by $\mathbf{N}$ and $\mathbf{B}$, and the plane spanned by $\mathbf{T}$ and $\mathbf{B}$ is called the *rectifying plane.* Now the curvature of the curve is, as we have seen, the rate of rotation, with respect to arc length, of the tangent line in the osculating plane. In three dimensions there is another important intrinsic function on the curve. Since $\mathbf{B}$ is a unit vector on Γ, $\langle \mathbf{B}', \mathbf{B} \rangle = 0$. Thus $\mathbf{B}'$ lies in the osculating plane. Since $\langle \mathbf{B}, \mathbf{T} \rangle = 0$, we have

$$\langle \mathbf{B}', \mathbf{T} \rangle + \langle \mathbf{B}, \mathbf{T}' \rangle = 0$$

Since $\mathbf{T}' = \kappa\mathbf{N}$, $\langle \mathbf{B}, \mathbf{T}' \rangle = \kappa \langle \mathbf{B}, \mathbf{N} \rangle = 0$, thus also $\langle \mathbf{B}', \mathbf{T} \rangle = 0$ so $\mathbf{B}'$ must be collinear with $\mathbf{N}$.

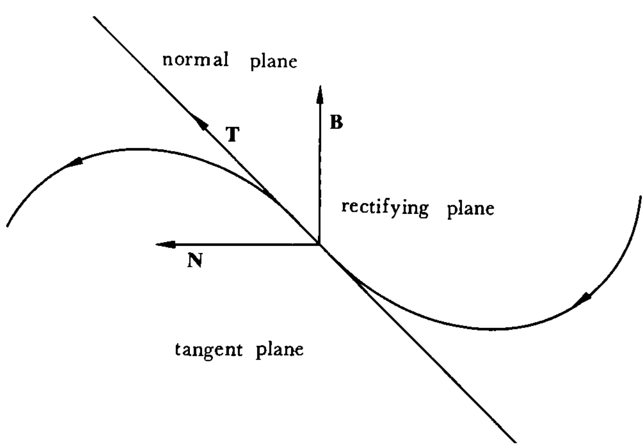


Figure 4.35

Definition 6. The **torsion** τ of a curve Γ is that function such that $\mathbf{B}' = -\tau\mathbf{N}$.

The torsion measures the torque, that is, the twisting of the osculating plane about the tangent line. That is, since the binormal is orthogonal to the osculating plane, the change in the binormal reflects adequately the change in the osculating plane. The Taylor development of the binormal in a neighborhood of a point $\mathbf{x}_0 = \mathbf{x}(0)$ is

$$\mathbf{B}(s) = \mathbf{B}(0) - \tau(0)\mathbf{N}(0)s + \varepsilon(s)$$

Thus (considering only first-order terms) the binormal at $\mathbf{x}(s)$ has moved $-\tau(0) \cdot s$ toward the normal. Thus if $\tau(0) > 0$, the osculating plane has twisted in the right-handed sense about the tangent line. At a point where $\tau = 0$, the osculating plane pauses; it may or may not change its direction of rotation about the tangent line. If $\tau \equiv 0$, the osculating plane remains fixed along the curve; it follows that the curve lies on this plane.

Proposition 6. *Let Γ be a curve in R^3. Γ is a plane curve if and only if $\tau \equiv 0$ along Γ.*

Proof. If Γ is a plane curve, let Π be the plane containing Γ. The tangent and normal to Γ always lie on Π, so the binormal is always the unit vector orthogonal to Π. Thus the binormal is constant, so $\mathbf{B}' \equiv 0$, thus $\tau \equiv 0$.

On the other hand, suppose $\tau \equiv 0$. Let $\mathbf{x} = \mathbf{x}(s)$ be the parametrization of Γ by arc length. Since $\tau \equiv 0$, $\mathbf{B}' \equiv 0$, so $\mathbf{B}$ is constant along Γ. If for some s_0, $\mathbf{x}(s_0)$ is not on the plane through $\mathbf{x}(0)$ and orthogonal to $\mathbf{B}$, then

$$\langle \mathbf{x}(s_0) - \mathbf{x}(0), \mathbf{B} \rangle \neq 0 \tag{4.41}$$

Let $\theta(s)$ be the function $\langle \mathbf{x}(s) - \mathbf{x}(0), \mathbf{B} \rangle$. Then $\theta'(s) = \langle \mathbf{T}(s), \mathbf{B} \rangle$ which is zero since $\mathbf{B} = \mathbf{B}(s)$ for all s and is orthogonal to $\mathbf{T}(s)$. Thus $\theta(s)$ is constant. Since $\theta(0) = 0$, $\theta(s_0) = 0$ also contradicting (4.41).

The fundamental formulas of space curve theory are those relating $\mathbf{T}'$, $\mathbf{N}'$, $\mathbf{B}'$ with $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$. We can now easily derive them.

Theorem 4.2. (Frenet–Serret Formula)

$$\begin{aligned}
\mathbf{T}' &= \phantom{-\kappa\mathbf{T}\quad} \kappa\mathbf{N} \\
\mathbf{N}' &= -\kappa\mathbf{T} \phantom{\kappa\mathbf{N}} + \tau\mathbf{B} \\
\mathbf{B}' &= \phantom{-\kappa\mathbf{T}\quad} -\tau\mathbf{N}
\end{aligned}$$

Proof. The first and the third are just the definitions of κ, τ, respectively. Since $\mathbf{N}$ is a unit vector, $\langle \mathbf{N}', \mathbf{N} \rangle = 0$, so $\mathbf{N}'$ lies in the rectifying plane. Write $\mathbf{N} = \alpha\mathbf{T} + \beta\mathbf{B}$; we must verify $\alpha = -\kappa$, $\beta = \tau$. But that follows from $\langle \mathbf{N}, \mathbf{T} \rangle = 0$, $\langle \mathbf{N}, \mathbf{B} \rangle = 0$. For

$$\alpha = \langle \mathbf{N}', \mathbf{T} \rangle = -\langle \mathbf{N}, \mathbf{T}' \rangle = -\kappa$$
$$\beta = \langle \mathbf{N}', \mathbf{B} \rangle = -\langle \mathbf{N}, \mathbf{B}' \rangle = -(-\tau) = \tau$$

Examples

36. The circular helix:

$$\mathbf{x}(t) = (a \cos t,\ a \sin t,\ bt)$$

We have already computed that $s = ct$, where $c = (a^2 + b^2)^{1/2}$, and

$$\mathbf{T}(s) = \frac{1}{c}\left(-a \sin\left(\frac{s}{c}\right),\ a \cos\left(\frac{s}{c}\right) b\right)$$

$$\kappa\mathbf{N}(s) = \frac{1}{c^2}\left(-a \cos\left(\frac{s}{c}\right),\ -a \sin\left(\frac{s}{c}\right), 0\right)$$

thus

$$\kappa = \frac{a}{c^2},\ \mathbf{N} = -\left(\cos\left(\frac{s}{c}\right), \sin\left(\frac{s}{c}\right), 0\right)$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{c}\left(-b \sin\left(\frac{s}{c}\right),\ b \cos\left(\frac{s}{c}\right),\ -a\right)$$

$$-\tau\mathbf{N} = \mathbf{B}' = \frac{1}{c^2}\left(-b \cos\left(\frac{s}{c}\right),\ -b \sin\left(\frac{s}{c}\right), 0\right)$$

thus $\tau = b/c^2$.

37. Let C be a curve in the xy plane, and let Γ be a curve of constant slope lying over the curve C (see Figure 4.36). Thus if $\mathbf{T}$ is the tangent to Γ, $\langle \mathbf{T}, \mathbf{E}_3 \rangle$ is constant. Let b be that constant. Then Γ has the parametrization

$$\mathbf{x}(t) = (x(t),\ y(t),\ b(t))$$

where $(x(t),\ y(t))$ parametrizes C. We may assume the parameter

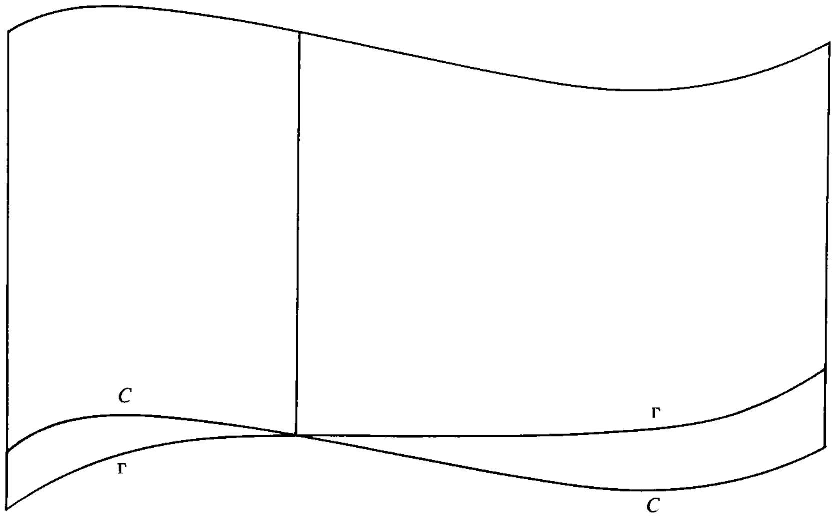


Figure 4.36

is arc length along C. Then

$$\mathbf{x}' = (x', y', b)$$

so

$$\frac{ds}{dt} = \|\mathbf{x}'\| = ((x')^2 + (y')^2 + b^2)^{1/2} = (1 + b^2)^{1/2}$$

Thus $s = (1 + b^2)^{1/2} t$ and the tangent to Γ is

$$\mathbf{T} = \frac{1}{(1 + b^2)^{1/2}} (x', y', b)$$

Thus

$$\kappa \mathbf{N} = \mathbf{T}' = \frac{1}{(1 + b^2)^{1/2}} (x'', y'', 0)$$

Now if κ_C is the curvature of C, since (x', y') is its tangent vector, $(-y', x')$ is its normal vector, so

$$(x'', y'') = \kappa_C(-y', x')$$

Thus

$$\kappa \mathbf{N} = \frac{\kappa_C}{(1 + b^2)^{1/2}} (-y', x', 0)$$

so

$$\kappa = \frac{\kappa_C}{(1 + b^2)^{1/2}} \qquad \mathbf{N} = (-y', x', 0)$$

Then

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{(1 + b^2)^{1/2}} (-bx', \ -by', (x')^2 + (y')^2)$$
$$= \frac{1}{(1 + b^2)^{1/2}} (-bx', \ -by', 1)$$

Differentiating,

$$-\tau \mathbf{N} = \mathbf{B}' = \frac{1}{(1 + b^2)^{1/2}} (-bx'', \ -by'', 0) = \frac{-b\kappa_C}{(1 + b^2)^{1/2}} (-y', x', 0)$$

Thus

$$\tau = \frac{b\kappa_C}{(1 + b^2)^{1/2}}$$

Local Behavior of a Curve

We shall now make a close study of the local behavior of a curve relative to the moving trihedron. Let Γ be a sufficiently differentiable curve, parametrized by arc length by $\mathbf{x} = \mathbf{x}(s)$, $-a < s < a$. We may perform a Euclidean transformation so that $\mathbf{x}(0) = 0$, $\mathbf{T}(0) = \mathbf{E}_1$, $\mathbf{N}(0) = \mathbf{E}_2$, $\mathbf{B}(0) = \mathbf{E}_3$. Expanding $\mathbf{x}(s)$ in a Taylor series, we obtain

$$\mathbf{x}(s) = \mathbf{x}(0) + \mathbf{x}'(0)s + \mathbf{x}''(0)\frac{s^2}{2} + \mathbf{x}'''(0)\frac{s^3}{6} + \varepsilon(s^3) \tag{4.42}$$

Now

$$\mathbf{x}' = \mathbf{T}, \mathbf{x}'' = \kappa \mathbf{N}, \mathbf{x}''' = \kappa' \mathbf{N} + \kappa \mathbf{N}' = \kappa' \mathbf{N} + \kappa(-\kappa \mathbf{T} + \tau \mathbf{B})$$

Evaluating these at zero and substituting into (4.42), we obtain

$$\mathbf{x}(s) = s\mathbf{E}_1 + \frac{\kappa}{2}s^2\mathbf{E}_2 + \frac{\kappa' s^3}{6}\mathbf{E}_2 - \frac{\kappa^2 s^3}{6}\mathbf{E}_1 + \frac{\kappa\tau s^3}{6}\mathbf{E}_3 + \varepsilon(s^3)$$

In coordinates,

$$x = s - \frac{\kappa^2 s^3}{6} + \varepsilon(s^3)$$

$$y = \frac{\kappa}{2}s^2 + \frac{\kappa'}{6}s^3 + \varepsilon(s^3)$$

$$z = \frac{\kappa\tau}{6}s^3 + \varepsilon(s^3)$$

Thus for small values of s, the given curve looks like the cubic curve given by the equations

$$y = \frac{\kappa}{2}x^2 \qquad z = \frac{\kappa\tau}{6}x^3 \qquad z^2 = \frac{4}{3}\frac{\tau^2}{\kappa}y^3$$

Figure 4.37 is a picture of this curve for $\kappa > 0$, $\tau > 0$. Notice that, so long as $\kappa\tau \neq 0$ the curve always passes through its osculating and normal planes, but lies on one side of its rectifying plane.

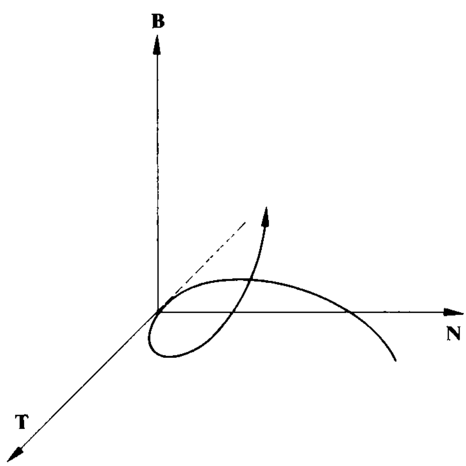


Figure 4.37

Now, just as the curvature determines plane curves up to a Euclidean motion, *space curves are so determined by the curvature and torsion.* The proof of this fact is by the same kind of application of Picard's existence and uniqueness theorem as we used in the case of the plane. We shall leave the verification to the interested reader.

Theorem 4.3 *Given continuous functions f, g defined in an interval I there is a space curve Γ: $\mathbf{x} = \mathbf{x}(s)$ given parametrically by arc length in some subinterval of I such that*

$$\kappa(s) = f(s) \qquad \tau(s) = g(s)$$

Γ is unique up to Euclidean motions in R^3.

● PROBLEMS

31. Show that a curve in R^3 is a plane curve if all its tangent planes pass through a given point.

32. Show that a curve in R^3 is a plane curve if its binormal is constant.

33. Let Γ be a curve in the plane and let γ be the intersection of the cylinder over Γ with the cone $x^2 + y^2 = z$, $z \geq 0$. Find the curvature of Γ in terms of that of γ. What is the torsion of γ?

34. Let Γ be a curve in space, and γ its projection onto the xy plane. What is the relation between the curvature and torsion of Γ and the curvature of γ?

35. Suppose that Γ is the intersection of the surface $z = y^2$ in R^3, with the plane $ax + by = 0$. What is the curvature of Γ at the origin?

36. Let Γ be the intersection of the surface $z = x^2 + 2y^2$ with the plane $ax + by = 0$. What are the curvature and torsion of Γ?

37. Let Γ be given in R^3 by $\mathbf{x} = \mathbf{x}(s)$. Let Σ be the surface swept out by the tangent lines to Γ. Show that a curve on Σ which is everywhere orthogonal to those tangent lines is given by

$$\mathbf{x} = \mathbf{x}(s) + (c - s)\mathbf{T}(s) \qquad \text{for some constant } c$$

4.5 Varying a Curve in the Plane

A **family of curves** in the plane is a collection of curves $\{\Gamma_c\}$, as c range through some set, usually of n-tuples of numbers. It is to be understood that the curves of the family vary smoothly; although we shall not make this idea precise. For example, if $x(t, c)$ are functions defined for real t and c lying in some set S, then the equations

$$x = x(t, c), y = y(t, c) \tag{4.43}$$

determine a family of curves: each curve in the family is found by fixing a value of c. We refer to Equations (4.43) as the **explicit form** of the family. More often, a curve is determined by a relation between x, y and a family could be given by an equation

$$F(x, y, c) = 0 \tag{4.44}$$

which, for fixed c gives the relation determining a curve. We refer to (4.44) as the **implicit form** of the family. Since it does not refer to any particular parametrization of the individual members, this form is particularly useful. The "constant" c which picks out the member of the family usually ranges through some set in R^n: in which case we refer to the family ((4.43), (4.44)) as an **n-parameter family of curves**.

Examples

38. A straight line in the plane is given by the equation

$$ax + by + c = 0 \tag{4.45}$$

Thus the set of all straight lines is given by (4.45) implicitly as a 3-parameter family of curves. If instead, we write down the slope-intercept form of a straight line,

$$y = mx + b \tag{4.46}$$

then we exhibit this family explicitly as a 2-parameter family of curves.

39. Let

$$x = x(t) \qquad y = y(t)$$

be the equation of a curve Γ in the plane, and consider the family of tangent lines to Γ. The equation of the line tangent to Γ at $(x(t), y(t))$ is

$$y = y(t) + \frac{y'(t)}{x'(t)}(x - x(t)) \tag{4.47}$$

This is the explicit form then of a 1-parameter family. (The parameter is t.)

40. Consider the case where Γ is the circle

$$x = \cos t \qquad y = \sin t$$

The family of tangent lines to Γ is given by the equation

$$y = \sin t - \frac{\cos t}{\sin t}(x - \cos t) \tag{4.48}$$

This simplifies to

$$y = -x \cot t + \csc t$$

We can make this appear even more palatable by taking $-\cot t$ as the parameter of the family. Letting $c = -\cot t$, we find $\csc t = -(1 + c^2)^{1/2}/c$, so (4.48) becomes

$$y = xc - \frac{(1 + c^2)^{1/2}}{c} \tag{4.49}$$

a 1-parameter family of lines.

41. Suppose a hoop is rolling along a horizontal line (see Figure 4.38). This collection of positions of the hoop forms a 1-parameter family of circles where the point of tangency with the horizontal (the

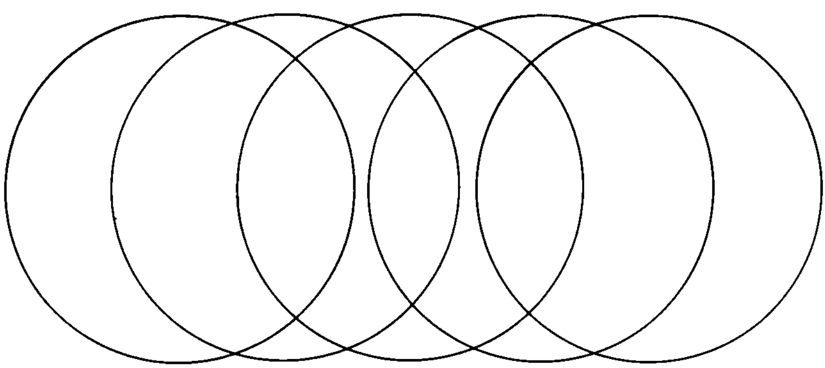


Figure 4.38

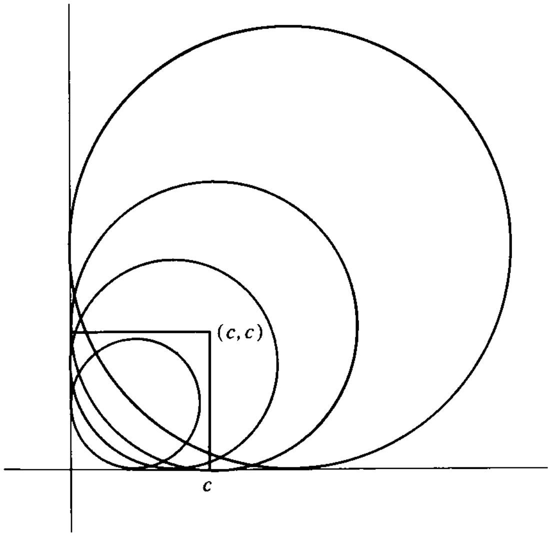


Figure 4.39

x axis) is taken to be the parameter. The implicit equation for the family is thus

$$(x - c)^2 + (y - 1)^2 = 1$$

42. The family of circles tangent to both the x axis and the y axis is a 1-parameter family of curves (Figure 4.39). We take for the parameter the point of tangency of the curve with the x axis. If r is the radius of the cth circle, then the equation of the family is clearly

$$(x - c)^2 + (y - r)^2 = r^2$$

It is easily seen that $r = c$; this follows from elementary geometric considerations. Thus the family is implicitly described by this equation

$$(x - c)^2 + (y - c)^2 = c^2 \tag{4.50}$$

43. The family of circles of radius 1 tangent to the parabola $y = x^2$ (Figure 4.40). We can take as the parameter the x coordinate c of the points of tangency. The center of the circle is on the line perpendicular to the parabola at (c, c^2). Thus if (r, s) are the coordinates

of the center of the cth circle, we have

$$s - c^2 = -\frac{1}{2c}(r - c)$$

$$(r - c)^2 + (s - c^2)^2 = 1$$

These equations have the solution

$$r = c + \frac{2c}{(1 + 4c^2)^{1/2}} \qquad s = c^2 - \frac{1}{(1 + 4c^2)^{1/2}}$$

Thus the implicit equation for this family of circles is

$$\left(x - c - \frac{2c}{(1 + 4c^2)^{1/2}}\right)^2 + \left(y - c^2 + \frac{1}{(1 + 4c^2)^{1/2}}\right)^2 = 1$$

44. Let Γ be a curve in the plane. We seek the family of tangents to Γ. If Γ is given as a function of arc length by $\mathbf{x} = \mathbf{x}(s)$, then the lines

$$\mathbf{g}(u) = \mathbf{x}(s) + u\mathbf{T}(s) \tag{4.51}$$

form the family of tangents to Γ with s as parameter. Suppose now that γ is a curve which is orthogonal to this family at every point. If $\mathbf{h}(s)$ is the point of intersection of γ with the particular tangent line

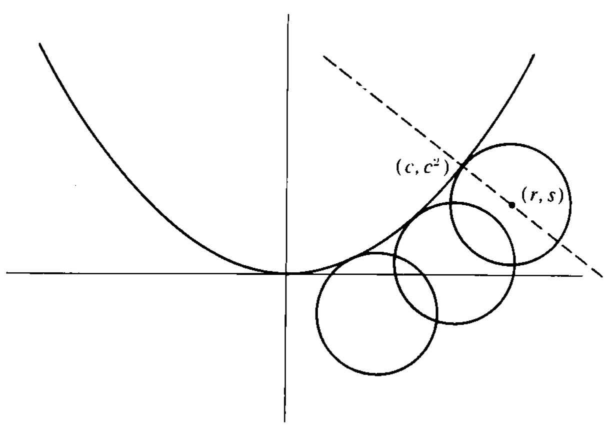


Figure 4.40

(4.51) at $\mathbf{x}(s)$, then γ is parametrized by $\mathbf{x} = \mathbf{h}(s)$. $\mathbf{h}(s)$ is then of the form (4.51) with a particular choice $u(s)$ of u. Writing then $\mathbf{h}(s) = \mathbf{x}(s) + u(s)\mathbf{T}(s)$, and differentiating, we obtain

$$\mathbf{h}'(s) = (1 - u'(s))\mathbf{T}(s) + u(s)\mathbf{T}'(s)$$

Since $\mathbf{h}'(s)$ is tangent to γ and thus, by assumption, orthogonal to $\mathbf{T}$, we must have $1 - u'(s) = 0$. Thus $u(s) = s + c$. So the family of curves orthogonal to the tangent lines to Γ is given by

$$\mathbf{x} = \mathbf{x}(s) + (s + c)\mathbf{T}(s) \tag{4.52}$$

The family of curves orthogonal to the tangents to the circle $z = e^{is}$ is given by

$$x = e^{is} + i(s + c)e^{is} = [1 + i(s + c)]e^{is}$$

These are just the evolutes of the circle.

The Differential Equation of a Family

A differential equation

$$y' = F(x, y)$$

determines a 1-parameter family of curves, if the function F is decent enough. For, under such conditions, for each c there is a unique solution of the initial value problem

$$y' = F(x, y) \qquad y(x_0) = c$$

The solution can be written $y = f(x, c)$, which can be considered as either the explicit, or implicit form of the family. Now, it is usually true that a 1-parameter family of curves *is* the family of solutions of some differential equation, and we would often like to find that differential equation.

Suppose, for example, that $y = f(x, c)$ is the equation of a given 1-parameter family. If $y = y(x)$ is one particular curve (i.e., $y(x) = f(x, c_0)$ for some fixed c_0), then these two equations must hold

$$y = f(x, c) \qquad y' = \frac{\partial f}{\partial x}(x, c)$$

for some value of c (i.e., $c = c_0$). It may be possible to eliminate the parameter c from these two equations, thus obtaining a relation between x, y, y' *which must be satisfied*; this is the differential equation of the family. For it is a differential equation which must be valid for each member of the family, and this is a differential equation which determines the family.

More generally, suppose the family is given implicitly by

$$F(x, y, c) = 0$$

If $x = x(t)$, $y = y(t)$ parametrizes one of the curves in the family, then there is a c such that

$$F(x(t), y(t), c) = 0 \tag{4.53}$$

identically in t. Differentiating now with respect to t, we have

$$\frac{\partial F}{\partial x}(x, y, c)x' + \frac{\partial F}{\partial y}(x, y, c)y' = 0 \tag{4.54}$$

If we can eliminate c from Equations (4.53) and (4.54), the result will be a relation between x, y, x', y' which must be satisfied for each curve in the family and thus is the differential equation of the family. Of course, if x is the parameter along the curve, and $y = y(x)$ is its equation, (4.54) becomes

$$\frac{\partial F}{\partial x}(x, y, c) + \frac{\partial F}{\partial y}(x, y, c)\frac{dy}{dx} = 0 \tag{4.55}$$

Examples

45. Consider the family of parabolas (Figure 4.41)

$$y^2 - cx = 0$$

Differentiating with respect to x (considering y as a function of x),

$$2yy' - c = 0$$

Thus the differential equation of the family is

$$y^2 - 2yy'x = 0$$

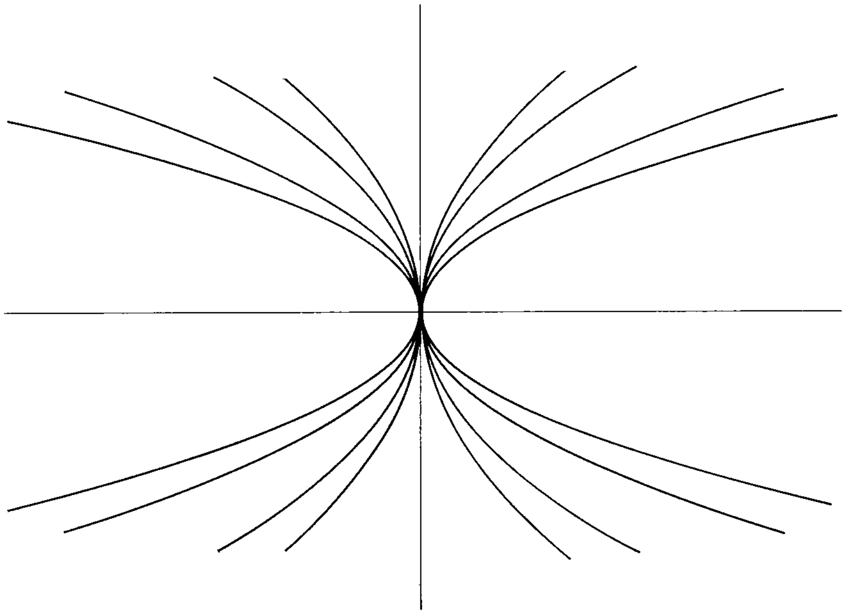


Figure 4.41

or, excluding the curve $y = 0$,

$$y - 2y'x = 0$$

46. The family $y = ce^x$ is given by the differential equation $y' = y$ (as we already know). The family $y = e^{cx}$ is given by the differential equation

$$y = \exp\left(\frac{y'}{y}x\right)$$

47. (Clairaut's Equation). Let $y = f(x)$ give a curve in the plane, and consider the family of lines tangent to that curve. That family is given implicitly (taking the x coordinate of the point of tangency as the parameter) by this equation,

$$y = f(x) + f'(c)(x - c) \tag{4.56}$$

Now, upon differentiation we find

$$y' = f'(c) \tag{4.57}$$

To say that we can eliminate c from the pair of Equations (4.56) and (4.57) amounts to saying that we can solve (4.57) for c as a function of y'. Then, upon eliminating we obtain as differential equation, the equation

$$y = y'x + h(y') \tag{4.58}$$

where $h(y')$ represents the expression $f(c) - f'(c)c$, considered as a function of y'.

Thus Equation (4.58), known as **Clairauts' equation**, is the general form of the differential equation of a family of lines tangent to a curve. Its solutions are

$$y = cx + h(c)$$

Notice that the given curve $y = f(x)$ also solves Equation (4.58) (because it is derived from (4.56) and (4.57) which hold under the substitution $y = f(x)$). It is called the **singular solution** of the equation.

48. The family of lines tangent to the parabola $y = x^2$ has the implicit form

$$y = c^2 + 2c(x - c) = 2cx - c^2$$

Differentiating, we obtain $y' = 2c$. Thus $c = \frac{1}{2}y'$, so we can eliminate c to obtain this differential equation of the family,

$$y = y'x - \tfrac{1}{4}(y')^2$$

49. The family of lines tangent to the circle $x^2 + y^2 = 1$ is given implicitly by

$$y = xc - \frac{(1 - c^2)^{1/2}}{2}$$

Then $y' = c$, so the Clairaut equation of the family is

$$y = y'x - \frac{(1 - (y')^2)^{1/2}}{2}$$

Family Orthogonal to a Given Family

50. Let F be a given family of curves. We propose to find a family G of curves everywhere orthogonal to F. Thus, if $\mathbf{p}$ is a point in the plane, and Γ is the curve in F through $\mathbf{p}$ with tangent $\mathbf{T}_1$, and γ is the curve in G through $\mathbf{p}$ with tangent $\mathbf{T}_2$ we must have $\langle \mathbf{T}_1, \mathbf{T}_2 \rangle = 0$. Suppose the family F is given by the differential equation (Figure 4.42)

$$a(x, y)x' + b(x, y)y' = 0 \tag{4.59}$$

Thus, since (x', y') is the tangent field to F, we must have $\mathbf{T}_2$ collinear with $(a(x,y),\ b(x,y))$ (for $\langle \mathbf{T}_1, (a, b) \rangle = 0$ by (4.59)). Thus the differential equation for the family G is

$$\frac{x'}{a(x, y)} = \frac{y'}{b(x, y)} \tag{4.60}$$

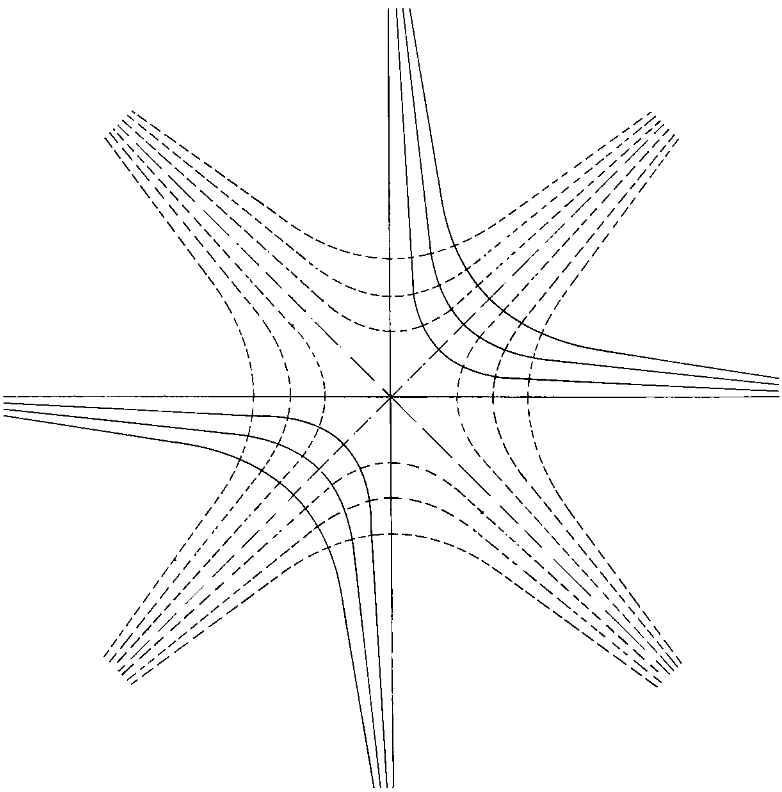


Figure 4.42

51. Find the family orthogonal to the family of hyperbolas

$$xy = c$$

The differential equation of this family is $yx' + xy' = 0$. Thus the differential equation of the orthogonal family is

$$\frac{x'}{y} = \frac{y'}{x}$$

or $xx' - yy' = 0$ which integrates to $x^2 - y^2 = c$.

52. The family orthogonal to the family of parabolas in Example 45 is given by the differential equation

$$\frac{1}{y} = \frac{y'}{-2x}$$

(here x is the parameter, so $x' = 1$). This integrates to

$$x^2 + \frac{y^2}{2} = c \tag{4.61}$$

53. Find the family which makes an angle of $\pi/4$ with the family (4.61). The differential equation of the family (4.61) is

$$2xx' + yy' = 0$$

The family orthogonal to this family has tangent collinear with $2x + iy$, thus the family we seek has tangent collinear with this vector rotated by $\pi/4$. Thus the tangent field is collinear with $e^{i(\pi/4)}(2x + iy)$, or, what is the same, $(1 + i)(2x + iy) = 2x - y + i(2x + y)$. Thus, the differential equation is

$$\frac{x'}{2x - y} = \frac{y'}{2x + y}$$

Envelopes

Many of the families we have been studying have the property that there is a curve (or curves) which is not a member of the family but bounds the family (see Figures 4.39–4.41). Similarly, for a family of lines tangent to a given curve, the curve bounds the family. Such a bounding curve is called an **envelope**. We want to see how to find envelopes for families.

First of all, some families do not admit envelopes. Clearly, the families $x = c$, $y = c$, $y = x^2 + c$ do not admit envelopes. However, if an envelope exists we can find it by the present techniques.

Definition 7. Let F be a family of curves in the plane. A curve Γ is an **envelope** for the family F, if through every point $\mathbf{p}$ in Γ there goes a curve in F which is tangent to Γ at $\mathbf{p}$.

Suppose that a family is given implicitly by

$$F(x, y, c) = 0$$

and that the curve Γ: $y = f(x)$ is an envelope of this family. Then, for every x_0 there is a $c(x_0)$ such that the curve C corresponding to $F(x, y, c(x_0)) = 0$ is tangent to Γ at $(x, f(x_0))$. Thus we must have

$$F(x_0, f(x_0), c(x_0)) = 0 \tag{4.62}$$

and since the curve C has the tangent direction $(1, f'(x_0))$, we must have, by (4.54),

$$\frac{\partial F}{\partial x}(x_0, f(x_0), c(x_0)) + \frac{\partial F}{\partial y}(x_0, f(x_0), c(x_0))f'(x_0) = 0 \tag{4.63}$$

Differentiating (4.62) with respect to x_0 we also find

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} f'(x_0) + \frac{\partial F}{\partial c} c'(x_0) = 0 \tag{4.64}$$

Comparing (4.63) and (4.64) we have as a result

$$\frac{\partial F}{\partial c}(x_0, f(x_0), c(x_0))\, c'(x_0) = 0 \tag{4.65}$$

Thus if (x, y) is on the evenlope Γ, there is a c such that

$$F(x, y, c) = 0 \qquad \frac{\partial F}{\partial c}(x, y, c) = 0$$

and we can eliminate c from this pair of equations to obtain an implicit equation of Γ. Notice that from (4.64), the equations

$$F(x, y, c) = 0 \qquad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y' = 0$$

also hold on Γ. Eliminating c from this pair we obtain once again the differential equation of the family, so the envelope must also satisfy this differential equation.

Examples

54. Find the envelopes of the family

$$(x - c)^2 + (y - 1)^2 = 1 \tag{4.66}$$

of Example 41. We differentiate with respect to c to find

$$-2(x - c) = 0 \quad \text{or} \quad x = c$$

Eliminating c we obtain $(y - 1)^2 = 1$, or $y = 2$, $y = 0$.

55. Find the envelopes of the family

$$(x - c)^2 + (y - c)^2 = c^2 \tag{4.67}$$

of Example 42. We must eliminate c from this equation and

$$-2(x - c) - 2(y - c) = 2c$$

or

$$x + y = c$$

Substituting this in (4.67) we obtain

$$(-y)^2 + (-x)^2 = (x + y)^2$$

or

$$2xy = 0$$

Thus the envelopes are $x = 0$, $y = 0$.

56. Find the envelopes of the family

$$y = x^2 \sin cx$$

Differentiation with respect to c yields

$$0 = cx^2 \cos cx$$

or

$$c = 0, \qquad cx = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

The condition $c = 0$ gives $y = 0$ which fails as an envelope. But $cx = \pi/2, 3\pi/2$ yields the envelopes $y = \pm x^2$ (Figure 4.43).

57. Find the envelope of the family given by

$$y = xy' + (1 + (y')^2)$$

This is a Clairaut equation and has the solution

$$y = cx + 1 + c^2$$

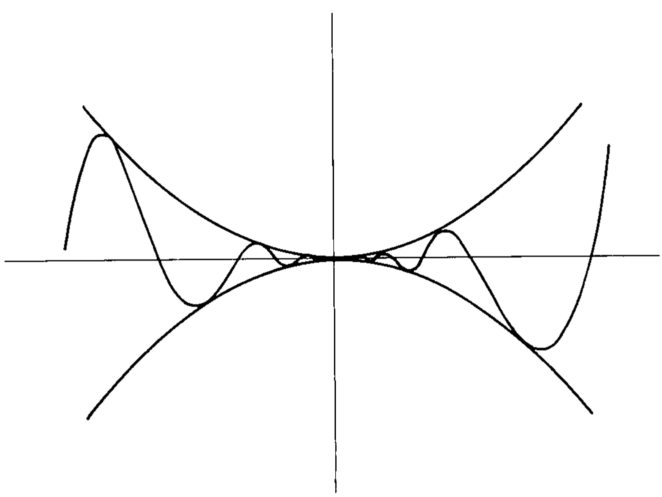


Figure 4.43

Differentiation with respect to c yields

$$0 = x + 2c \quad \text{or} \quad c = \frac{x}{2}$$

Thus the envelope of this family is the curve

$$y = \frac{3x^2}{4} + 1$$

• EXERCISES

12. Find the differential equations for these families of curves:
 (a) $xyc = 1$ (c) $xe^{cy} = 1$
 (b) $\sin xy - a \cos xy = 0$ (d) $x \sin y + c \sin x = 0$
 (e) $ye^{c(x+y)} = 1$
 (f) $\sin(x + y + c) + \cos(x + y + c) = 1$

13. Find the implicit form of the family given by these differential equations:
 (a) $xy' - yx' = 0$ (c) $(y')^2 + y^2 = 1$
 (b) $x' + yy' = 1$ (d) $y + y'x + \sin y = 0$
 (e) $y'(\sec x - \tan x) = 1 - y$

14. Find the implicit form and the differential equation of the family of circles with center on the y axis and tangent to the x axis.

15. Find the family of ellipses with foci at $(-1, 0)$, $(0, 1)$.

16. Find the family of curves orthogonal to the family in Exercise 14; Exercise 15.

17. Find the family orthogonal to the families of Exercises 12(a), (b), (f), 13(b), (d).

18. Find the family making an angle of $\pi/3$ with the family of Exercises 12(a), (b), (c).

19. Find the envelopes of the families of Exercises 12(a), (b), (c), (d), (e).

20. Find the envelopes of these families:
 (a) $y = \sin(x - c)^2$ (c) 13e
 (b) 13b (d) $y = e^x \sin cx$
 (e) The family of cardiods $r = (1 + c)^{-1}(1 + c \cos \theta)$.
 (f) The family $r = \sin a\theta$.

• PROBLEMS

38. Find the family of evolutes of the parabola $y = x^2$. Find the family orthogonal to this family of evolutes.

39. Find the family orthogonal to the family of spirals $r = ce^{\theta}$.

40. A ladder 10 feet tall originally leaning against a building slips (Figure 4.44). Find the family of curves which are the trajectories of the points on the ladder.

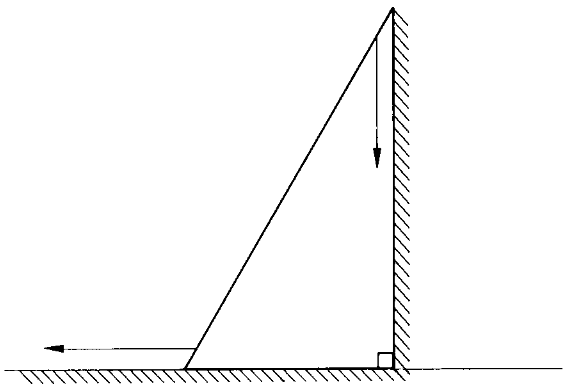


Figure 4.44

41. Find the family of trajectories of the points on the circumference of a ball rolling on a horizontal plane.

42. A line segment of length 2 has its endpoints on the parabola $y = x^2$. Find the trajectories of points x_0 on the segment as it slides along the parabola (Figure 4.45).

43. A ball of unit mass is at the end of a string of unit length attached to the top of a vertical bar rotating at constant angular velocity. Find the path of motion of the ball assuming its position and velocity at time $t = 0$ to be $(1, 0, 0)$, $(1, 0, 1)$, respectively. Find the trajectory of any point on the string.

44. Find the family of curves swept out by the midpoints of bars of given length with endpoints along the curve $xy = 1$ in the first quadrant.

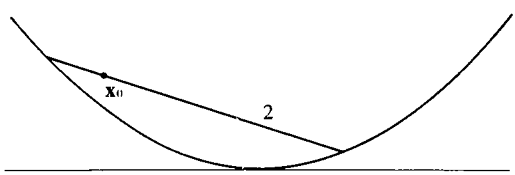


Figure 4.45

4.6 Vector Fields and Fluid Flows

We have come across vector fields several times already: the gradient of a function, the gravitational field, a field of forces, are all vector fields. We now want to study such fields in connection with fluid flows: motions of a mass of noninterreacting particles.

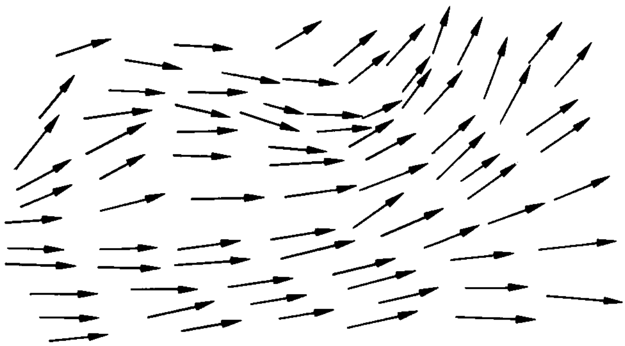


Figure 4.46

A **vector field** is a function which assigns to each point in a given domain in R^n, a vector in R^n, usually considered as based at the given point. Thus, a vector field defined on D in R^n is nothing more than an R^n-valued function on U, but interpreted pictorially as in Figure 4.46.

Examples

58. A body in space sets up a field of gravitational attraction. Suppose there is a body of unit mass situated at the origin. According to Newton's laws another body of unit mass is attracted to the given body at the origin with a force proportional to the inverse of the distance squared. We represent this attraction at a point $\mathbf{p}$ by a vector directed toward the origin and of length $\|\mathbf{p}\|^{-2}$ (see Figure 4.47). Thus the gravitational field of a body situated at the origin is the vector field defined on $R^3 - \{0\}$ by

$$\mathbf{v}(\mathbf{p}) = \frac{-\mathbf{p}}{\|\mathbf{p}\|^3}$$

or, in rectangular coordinates

$$\mathbf{v}(x, y, z) = -\frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}$$

59. Given a family of curves, we may consider the field of unit tangents to the family (Figure 4.48). In particular the field of tangents to the family of circles $x^2 + y^2 = c^2$ is defined on $R^2 - \{0, 0\}$,

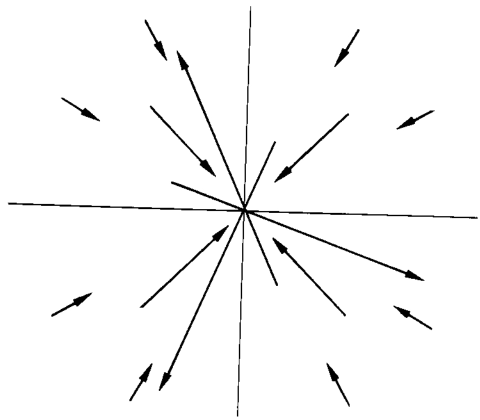


Figure 4.47

and is given by

$$\mathbf{T}(x, y) = \frac{(-y, x)}{(x^2 + y^2)^{1/2}}$$

The family of unit tangents to the family of rays is defined on $R^2 - \{0, 0\}$ by

$$\mathbf{T}(x, y) = \frac{(x, y)}{(x^2 + y^2)^{1/2}}$$

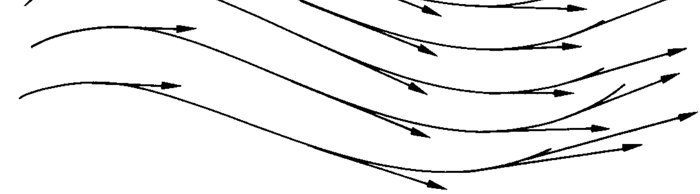


Figure 4.48

If we are given a vector field $\mathbf{v}$ on a domain D in R^n, the questions arise: Is it a field tangent to a family of curves, and if so, can we discover the curves?

Suppose then that $\mathbf{v}$ is a given vector field in the domain D, and Γ is a curve in D such that $\mathbf{v}(\mathbf{x})$ is tangent to Γ at each point $\mathbf{x}$ on Γ. Let $\mathbf{f}$ be a function which parametrizes the curve Γ. Then $\mathbf{f}'(t)$ is tangent to Γ at $\mathbf{f}(t)$ so we must have $\mathbf{f}'(t)$ and $\mathbf{v}(\mathbf{f}(t))$ collinear. In particular then, if $\mathbf{f}$ is a solution of the differential equation

$$\mathbf{f}'(t) = \mathbf{v}(\mathbf{f}(t))$$

then $\mathbf{f}$ parametrizes a curve tangent to the given field. In the terminology of the preceding section

$$\frac{d\mathbf{x}}{dt} - \mathbf{v}(\mathbf{x}) = \mathbf{0}$$

is the (parametric) differential equation of the family of curves tangent to the vector field.

60. Suppose $\mathbf{v}(x, y) = (x, 2y)$. (Figure 4.49.) Then the family of curves tangent to the vector field $\mathbf{v}$ is given parametrically by this differential equation:

$$x' = x \qquad y' = 2y \tag{4.68}$$
$$x(0) = x_0 \qquad y(0) = y_0$$

The solution is given by

$$x = x_0 e^t \qquad y = y_0 e^{2t} \tag{4.69}$$

We can write this family of curves implicitly as

$$y - cx^2 = 0$$

(taking the constant c as $y_0 x_0^{-2}$). Thus the family we seek is a system of parabolas.

Another way to find the implicit equation of the curve is to divide one equation in (4.68) by the other:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2y}{x}$$

This we can solve directly by separation of variables.

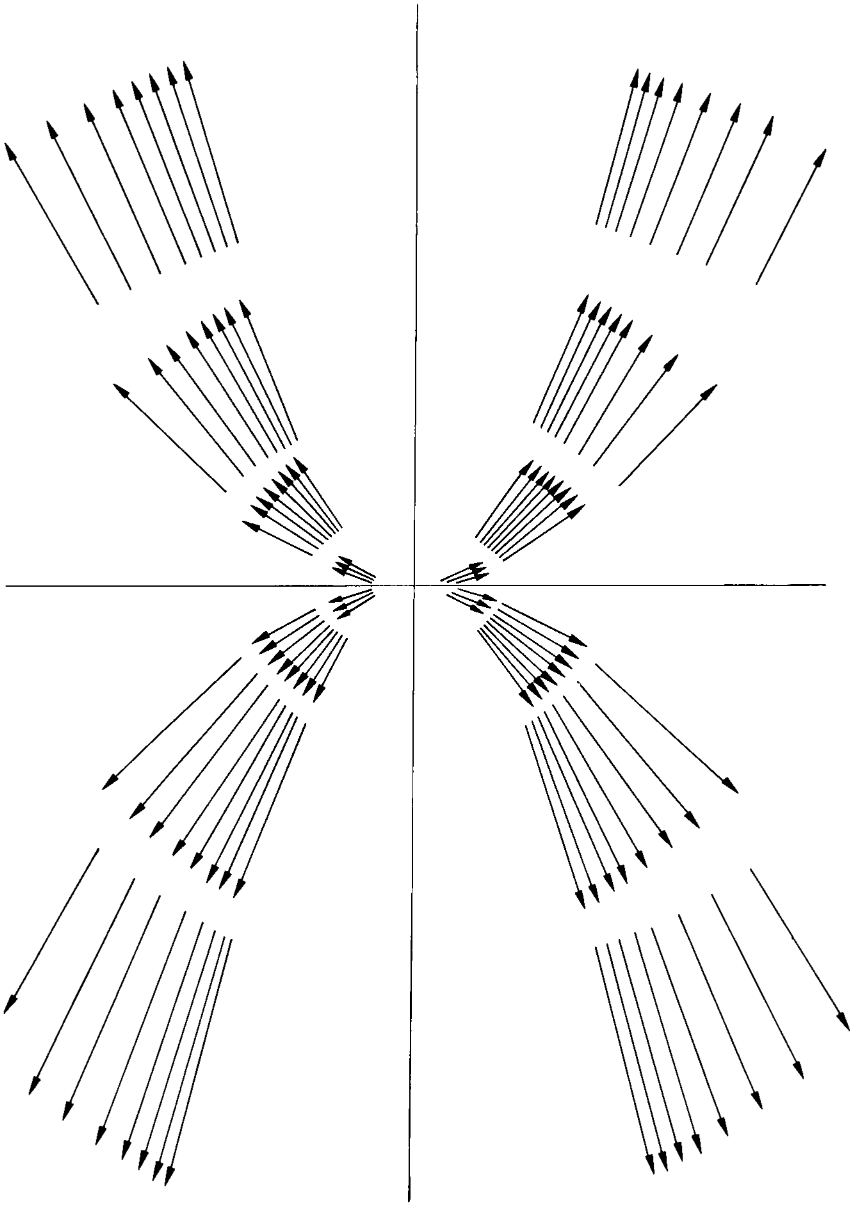


Figure 4.49

61. Let $\mathbf{v}(x, y) = (x + y, 1)$. Then the differential equation is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x+y}{1} \quad \text{or} \quad y' = x + y$$

which has the general solution

$$y = -(x + 1) + ce^x$$

Now let us consider a fluid in motion in a domain D in R^n. The equations of fluid motion are written as follows. We suppose that at time $t = 0$ there is a particle of fluid at each point $\mathbf{x}_0$ in D. The position of that particle at the subsequent time t is denoted by $\phi(\mathbf{x}_0, t)$. The equation of motion then is

$$\mathbf{x} = \phi(\mathbf{x}_0, t) \tag{4.70}$$

For a fixed $\mathbf{x}_0$, the curve described by (4.69) is the path of the particle which is at $\mathbf{x}_0$ at time $t = 0$. Thus we are assuming that

$$\mathbf{x}_0 = \phi(\mathbf{x}_0, 0) \tag{4.71}$$

It is also assumed that no two particles can ever occupy the same position at the same time. Then for each t, the function $\phi(\mathbf{x}_0, t)$ is one-to-one and thus can always be inverted: there is also a function $\psi(\mathbf{x}, t)$ which describes the $t = 0$ position of the particle at $\mathbf{x}$ at time t such that

$$\mathbf{x} = \phi(\mathbf{x}_0, t) \quad \text{if and only if} \quad \mathbf{x}_0 = \psi(\mathbf{x}, t)$$

Definition 8. Given the fluid motion described by Equation (4.70) its **velocity at the time** $t = t_0$ is the vector field

$$\left.\frac{\partial \phi}{\partial t}\right|_{t=t_0}$$

situated at the point $\mathbf{x} = \phi(\mathbf{x}_0, t_0)$. If the vector field is independent of time, we say that the fluid motion is a **steady flow**.

Thus the velocity field of a flow at time t_0 and point $\mathbf{x}$ is the velocity $\mathbf{v}(\mathbf{x}, t_0)$ of the particle which is at $\mathbf{x}$ at that time. If the velocity is independent of the time, or the particular particle, the flow is steady. The flow in a river of constant volume is determined by the shape of the river bed, and thus tends to be steady, whereas the flow of clouds in the sky is time dependent. If the flow is steady, then the path lines (the curves described by (4.70)) are the curves of the family tangent to the velocity field. If the velocity field is time dependent, then these tangent families (called **the lines of force**) vary with time and have little to do with the paths of individual particles. This is easy to see. Suppose the flow

$$\mathbf{x} = \phi(\mathbf{x}_0, t) \tag{4.72}$$

has the velocity field $\mathbf{v}(\mathbf{x}, t)$. Then the path lines (4.72) are the solutions of the differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}(t), t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{4.73}$$

The lines of force at time $t = t_0$ are the solutions of the equation

$$\frac{d\mathbf{x}}{d\tau} = \mathbf{v}(\mathbf{x}(\tau), t_0) \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{4.74}$$

These are the same differential equations if and only if $\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t_0)$ for all t, that is, if and only if the flow is steady.

Examples

62. Consider the flow in R^2 given by Equation (4.67):

$$x = x_0 e^t \qquad y = y_0 e^{2t} \tag{4.75}$$

Then

$$x' = x_0 e^t \qquad y' = 2y_0 e^{2t} \tag{4.76}$$

Thus the velocity at time t of the particle originally at (x_0, y_0) is $(x_0 e^t, 2y_0 e^{2t})$. To find the velocity field we must solve (4.75) for x_0, y_0 in terms of $\mathbf{x}$, t and substitute. Thus (4.76) becomes

$$x' = x \qquad y' = 2y$$

so the velocity field is $\mathbf{v}(x, y) = (x, 2y)$ and the flow is steady.

63. Consider the flow in R^3 given by

$$x = x_0 + t \qquad y = y_0 + t^2 \qquad z = z_0 + t^3$$

$$x' = 1 \qquad y' = 2t \qquad z' = 3t^2$$

Thus the velocity field

$$\mathbf{v}(x, y, z) = (1, 2t, 3t^2)$$

is independent of position but is time dependent. In fact, the path lines are independent of position and are just translates of the twisted cubic (Figure 4.50). It is as if all of space were being rigidly translated along the line curve $y = x^2$, $z = x^3$. Notice that since the velocity field at any given time $t = t_0$ is a constant field, the lines of force are straight lines.

64. $x = x_0 + t$, $y = y_0(1 + t)$, $z = z_0 e^t$. Then

$$\left.\frac{\partial \mathbf{x}}{\partial t}\right|_t = (1, y_0, z_0 e^t)$$

so the velocity field is

$$\mathbf{v}(x, y, z) = \left(1, \frac{y}{1+t}, z\right)$$

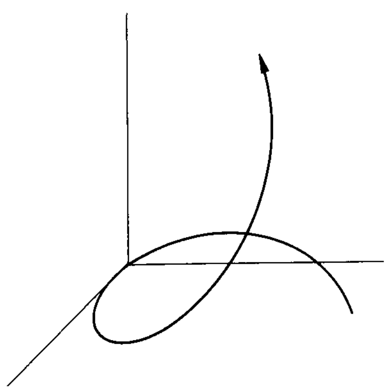


Figure 4.50

the flow is not steady. The lines of force at time $t = t_0$ are the solutions of

$$x' = 1 \qquad y' = \frac{y}{1 + t_0} \qquad z' = z$$

so is the family

$$x = x_0 + t \qquad y = y_0 \exp\left(\frac{t}{1 + t_0}\right) \qquad z = z_0 e^t$$

which is quite different from the family of path lines.

65. The flow is given by

$$x = x_0 e^t \qquad y = y_0 e^{-t} + x_0(e^t - e^{-t}) \qquad z = z_0 e^{2t} - x_0(e^t - e^{2t}) \tag{4.77}$$

$$x' = x_0 e^t \qquad y' = -y_0 e^{-t} + x_0 e^t + x_0 e^{-t}$$
$$z' = 2z_0 e^{2t} - x_0(e^t - e^{2t}) \tag{4.78}$$

The Equations (4.77) are linear in x_0, y_0, z_0, so we may solve for these in terms of x, y, z. Doing so, and substituting the result in (4.78), we obtain the velocity field of the flow,

$$\mathbf{v}(x, y, z) = (x, 2x - y, 2z + x)$$

This flow is time independent, or steady.

It is an immediate consequence of the uniqueness assertion of Picard's theorem that a flow is completely determined by its velocity field. For the flow equation is the solution of the initial value problem (4.73), which is unique. Notice also that the existence part of Picard's theorem asserts that there always is a flow associated with a given velocity field (which is sufficiently smooth).

The last remark we care to make at this time (we shall continue the study of fluid flows in Chapter 8) is that in the case of a steady flow, the particles follow one another along a fixed family of paths (whereas in general each particle determines its own path). These are of course the lines of force.

What we must show is this: If two particles $\mathbf{x}_0$, $\mathbf{x}_1$ occupy the same position at different times (of course), then they follow the same paths. That is, if there are s_0, s_1 such that

$$\phi(\mathbf{x}_0, s_0) = \phi(\mathbf{x}_1, t_1)$$

then the curves

$$\Gamma_0 : \mathbf{x} = \phi(\mathbf{x}_0, t) \qquad \Gamma_1 : \mathbf{x} = \phi(\mathbf{x}_1, t)$$

are the same, except for parametrization. The following proposition proves this, and more. It makes explicit the relation between the two parametrizations.

Proposition 7. *Suppose* $\mathbf{x} = \phi(\mathbf{x}_0, t)$ *describes a steady flow. If for some* $(\mathbf{x}_0, s_0)$, $(\mathbf{x}_1, s_1)$, *we have*

$$\phi(\mathbf{x}_0, s_0) = \phi(\mathbf{x}_1, s_1)$$

then

$$\phi(\mathbf{x}_0, s_0 + t) = \phi(\mathbf{x}_1, s_1 + t) \qquad \textit{for all } t \tag{4.79}$$

In particular, $\mathbf{x}_1 = \phi(\mathbf{x}_0, s_0 - s_1)$.

Proof. The proof is simply that the two functions in (4.79) solve the same initial value problem. Let $\mathbf{v}(\mathbf{x})$ be the velocity field of the flow (by assumption $\mathbf{v}$ is time independent). Consider these functions

$$\mathbf{f}_0(t) = \phi(\mathbf{x}_0, s_0 + t) \qquad \mathbf{f}_1(t) = \phi(\mathbf{x}_1, s_1 + t) \tag{4.80}$$

We have

$$\mathbf{f}_0(0) = \mathbf{f}_1(0)$$

Since

$$\frac{\partial \phi}{\partial t}(\mathbf{x}_0, t) = \mathbf{v}(\phi(\mathbf{x}_0, t))$$

for all $\mathbf{x}$, t, we have

$$\frac{d\mathbf{f}_0}{dt}(t) = \frac{\partial}{\partial t}\phi(\mathbf{x}_0, s_0 + t) = \mathbf{v}(\phi(\mathbf{x}_0, s_0 + t)) = \mathbf{v}(\mathbf{f}_0(t))$$

$$\frac{d\mathbf{f}_1}{dt}(t) = \mathbf{v}(\mathbf{f}_1(t))$$

Thus $\mathbf{f}_0, \mathbf{f}_1$ solve the same first-order differential equation and by (4.80) have the same value at 0. Thus $\mathbf{f}_0 = \mathbf{f}_1$ identically.

Planetary Motion

We conclude this chapter with a study of the classical equations of planetary motion. This study first requires these simplifications. We assume all action is in a plane, and that the only force acting is that due to the sun's gravitational field. These simplifications approximate the true situation with enormous accuracy. For the other forces acting on the body are gravitational forces due to other celestial bodies, which are either too far away or too small, relative to the sun, to make a substantial contribution. According to Newton's laws, the acceleration of a body due to the gravitational field is proportional to the field. The motion is thus completely determined by this force and an initial position and velocity. For if $s = s(t)$ is the equation of motion, then s is the solution of an initial value problem;

$$\begin{aligned} s(0) &= s_0 \\ s'(0) &= v_0 \\ s''(t) &= kF(s(t)) \end{aligned}$$

where F is the given force field.

Our purpose here is to describe the motion of planets in terms of an observed position and velocity. If we locate the sun at the origin, then the gravitation force field is given (in complex notation) by

$$F(z) = -\frac{z}{|z|^3}$$

Thus, we must explicitly solve this system

$$\begin{aligned} z(0) &= z_0 \\ z'(0) &= v_0 \\ z''(t) &= -\frac{z(t)}{|z(t)|^3} \end{aligned} \tag{4.81}$$

The best way to solve this is by means of polar coordinates. Write $z(t) = r(t)e^{i\theta(t)}$. Then differentiating, we have

$$z' = r'e^{i\theta} + i\theta' re^{i\theta} \tag{4.82}$$

$$z'' = r''e^{i\theta} + 2i\theta' r'e^{i\theta} + i\theta'' re^{i\theta} - (\theta')^2 re^{i\theta} \tag{4.83}$$

and Equation (4.81) becomes

$$z'' = r''e^{i\theta} + 2i\theta' r'e^{i\theta} + i\theta'' re^{i\theta} - (\theta')^2 re^{i\theta} = -\frac{e^{i\theta}}{r^2} \tag{4.84}$$

Multiplying through by $e^{-i\theta}$, we obtain

$$r'' - (\theta')^2 r + i(2\theta' r' + r\theta'') = \frac{-1}{r^2}$$

which reduces to this system (equating real and imaginary parts):

$$r'' - (\theta')^2 r = \frac{-1}{r^2} \qquad 2\theta' r' + r\theta'' = 0 \tag{4.85}$$

The second equation reads

$$2(\ln r)' = \frac{2r'}{r} = \frac{\theta''}{\theta'} = (\ln \theta')'$$

so either $\theta' = 0$ or θ' is proportional to r^{-2}. We have then these two alternatives. In one case θ is constant, in which case the planet approaches the sun along a straight line. In the other case, the planet rotates around the sun at an angular velocity inversely proportional to the square of the distance from the sun (the closer it is to the sun the faster it rotates around it). Notice also that the solution $r =$ constant, $\theta' =$ constant is possible, so that an admissible path is that of circular motion of constant angular velocity. The angular velocity decreases as the circle gets larger.

We proceed now to the full solution of (4.84). We already have $\theta' r^2 = h$, a constant (determined by the initial conditions). From (4.84), we obtain

$$z'' = -\frac{e^{i\theta}}{r^2} = -\frac{1}{h} e^{i\theta}\theta' = \frac{i}{h}(e^{i\theta})'$$

Thus we can integrate to obtain

$$z' = \frac{i}{h} e^{i\theta} + C$$

Where $C = \rho e^{i\omega}$ is an arbitrary constant. Now, using (4.82) we have

$$r'e^{i\theta} + i\theta' r e^{i\theta} = \frac{i}{h} e^{i\theta} + \rho e^{i\omega}$$

Multiply through by $e^{-i\theta}$ and equate imaginary parts:

$$\theta' r = \frac{1}{h} + \rho \sin(\omega - \theta)$$

Once again using $\theta' r^2 = h$, we obtain this implicit relation between r and θ:

$$h = r\left(\frac{1}{h} + \rho \sin(\omega - \theta)\right)$$

or

$$r = \frac{h^2}{1 + \rho h \sin(\omega - \theta)} \tag{4.86}$$

The constants ρ, h, ω are to be determined by the initial conditions. Equation (4.86) is the polar form of the equation of a conic with one focus at the origin. If $\rho h < 1$, it is an ellipse; $\rho h = 1$, a parabola; and $\rho h > 1$, a hyperbola. These are then the possible paths of motion of a planet, or comet, around the sun.

• EXERCISES

21. Find the family of curves tangent to the given vector fields:
    (a) $\mathbf{v}(x, y) = (x, -y)$
    (b) $\mathbf{v}(x, y) = (-y, x)$
    (c) $\mathbf{v}(x, y, z) = (-x, -y, z)$
    (d) $\mathbf{v}(x, y, z) = (x, -1, z)$
22. Find a field of vectors tangent to these families:
    (a) $z = e^{(1+ci)t}$
    (b) $z = e^{c+it}$
    (c) $x = 2ct$, $y = 1 - (ct)^2$
    (d) $x = x_0 + t$, $y = e^t y_0$, $z = \sin t$

23. Find the velocity field of these flows:
   (a) $\mathbf{x}(t) = (e^{-t}x_0, y_0 + t, e^{-t}z_0)$
   (b) $\mathbf{x}(t) = (x_0(1+t), y_0(1+t), z_0 + t^2(x_0^2 + y_0^2))$
   (c) $\mathbf{x}(t) = (x_0, y_0 + t, z_0 \cos t)$
   (d) $\mathbf{x}(t) = e^{-t}(x_0, y_0, z_0 \cos t)$
24. Find the flow with the given velocity field:
   (a) $\mathbf{v}(t) = t(x, y, z)$ (c) Exercise 21(b).
   (b) $\mathbf{v}(t) = t(-y, x, 1)$ (d) Exercise 21(c).
25. Is there a steady flow whose path lines are the trajectories of the particles at $(x_0, y_0, 0)$ at time $t = 0$ in the flow in Exercise 23(b)?

• PROBLEMS

45. Under what conditions on the velocity field of a flow are the lines of force at all times the same as the paths of motion?

46. Consider a flow which spirals around the line L: $x = y = z$ at constant angular velocity, whose distance from the origin increases exponentially with time and whose distance from L decreases exponentially with time. Find the velocity field of the flow.

47. If we are given a family of curves in the plane we may consider the tangent field of the family as well as its differential equation and the tangent field of the orthogonal family as well as its differential equation. How are all these formulas related?

4.7 Summary

The image in R^n of an interval under a one-to-one C^1 function with a nowhere vanishing derivative is called a *curve*. If Γ is a curve given by the function

$$\mathbf{x} = \mathbf{f}(t) \qquad a \le t \le b$$

the variable t is called the *parameter* of the curve. If

$$\mathbf{x} = \mathbf{g}(\tau) \qquad \alpha \le \tau \le \beta$$

is another parametrization of the curve, there is a one-to-one function

$$t = \sigma(\tau)$$

mapping the interval $[\alpha, \beta]$ onto the interval $[a, b]$ such that

$$\mathbf{g}(\tau) = \mathbf{f}(\sigma(\tau)) \qquad \alpha \le \tau \le \beta$$

If $\sigma' > 0$ (σ is increasing) we say that the parameters t, τ induce the same *orientation* on Γ. This notion divides all parametrizations into two classes. An *oriented curve* is one for which one of these classes, the *well-oriented parameters* is chosen.

If F is a differentiable function of two variables such that ∇F is never zero, then the equation $F(x, y) = 0$ defines a curve *implicitly*. For we can find a parametrization

$$x = f(t) \qquad y = g(t)$$

for the set $F(x, y) = 0$. Similarly, if F, G are two differentiable functions of three variables such that ∇F and ∇G are everywhere independent in the set

$$F(x, y, z) = 0 \qquad G(x, y, z) = 0$$

implicitly defines a curve in R^3.

If Γ is an oriented curve with a parametrization

$$\mathbf{x} = \mathbf{f}(t) \qquad a \leq t \leq b$$

the length of Γ *between* $\mathbf{f}(a)$ *and* $\mathbf{f}(t)$ for $a \leq t \leq b$ is defined to be the least upper bound of all sums

$$\sum_{i=1}^{k} \|\mathbf{f}(t_i) - \mathbf{f}(t_{i-1})\|$$

over all choices of points $t_0, \ldots, t_k$ such that

$$a = t_0 \leq t_1 \leq \cdots \leq t_k = t$$

If $s(t)$ is this number, the function $s = s(t)$, $a \leq t \leq b$ gives a parametrization of Γ. This is the parametrization by arc length. s is the solution of the differential equation

$$s'(t) = \|\mathbf{f}'(t)\|$$
$$s(a) = 0$$

The *unit tangent* to a curve $\Gamma\colon \mathbf{x} = \mathbf{x}(s)$ is the vector $\mathbf{T}(s) = \mathbf{x}'(s)$. The tangent line is the line through $\mathbf{f}(s)$ spanned by this vector. The *unit normal* to the curve is a choice of unit vector $\mathbf{N}(s)$ lying on the line spanned by $\mathbf{T}'(s)$. In two dimensions $\mathbf{N}$ is chosen so that the rotation $\mathbf{T} \to \mathbf{N}$ is counterclockwise. In three dimensions the $\mathbf{T} - \mathbf{N}$ plane is called the osculating plane.

The *unit binormal* is the vector $\mathbf{B}$ so that the basis $\mathbf{T} \to \mathbf{N} \to \mathbf{B}$ is a right-handed orthornormal basis: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. This frame is determined by these differential equations, *the Frenet-Serret formulas*:

$$\mathbf{T}' = \kappa\mathbf{N}$$
$$\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}$$
$$\mathbf{B}' = -\tau\mathbf{N}$$

The scalar functions κ, τ, the *curvature* and *torsion* respectively of the curve are defined by the first and third equations. The curvature κ is the angular velocity of the tangent in the osculating plane and the torsion is the angular velocity of the osculating plane about the tangent. A curve in R^3 is uniquely determined (but for Euclidean motions) by its curvature and torsion. A curve in R^2 is uniquely determined (but for Euclidean motion) by its curvature.

If $\mathbf{x} = \mathbf{f}(t)$ is the equation of motion of a particle, we call the curve described by this function *the path of motion.* ds/dt is the *speed*, $\mathbf{f}'(t)$ is the *velocity* and $\mathbf{f}''(t)$ is the acceleration of the particle. The acceleration vector lies in the osculating $(\mathbf{T} - \mathbf{N})$ plane. We can write

$$\text{acceleration} = a_T\mathbf{T} + a_N\mathbf{N}$$

where a_T is the *tangential acceleration* and a_N the *normal acceleration*. These equations hold:

$$a_T = \frac{d^2s}{dt^2} \qquad a_N = \left(\frac{ds}{dt}\right)^2\kappa$$

where κ is the curvature of the path of motion.

A *family of curves in the plane* is a collection of curves $\{\Gamma_c\}$ as c ranges through some set. A pair of equations

$$x = x(t, c) \qquad y = y(t, c)$$

determines a family of curves. This is the *explicit form* of the family. A functional equation

$$F(x, y, c) = 0$$

also determines a family. This is the *implicit form* of the family. The set of solutions of a differential equation

$$a(x, y)x' + b(x, y)y' = 0 \tag{4.87}$$

forms a family of curves in the plane. If

$$F(x, y, c) = 0 \tag{4.88}$$

is the implicit form of a family its differential equation is found by eliminating c from (4.88) and

$$\frac{\partial F}{\partial x} x' + \frac{\partial F}{\partial y} y' = 0$$

If (4.87) is the differential equation of a family F, the family of curves orthogonal to the family F is given by the differential equation

$$-b(x, y)x' + a(x, y)y' = 0$$

A *vector field* in a domain $U \subset R^n$ is an R^n-valued function defined in U. The vector associated to a particular point in U is depicted as originating at that point. A fluid flow is given by the function

$$\mathbf{x} = \boldsymbol{\phi}(\mathbf{x}_0, t)$$

with these properties:

(i) $\boldsymbol{\phi}(\mathbf{x}_0, 0) = \mathbf{x}_0$.
(ii) $\boldsymbol{\phi}$ has continuous partial derivatives.
(iii) For each t the function $\mathbf{x} = \boldsymbol{\phi}(\mathbf{x}_0, t)$ is invertible.

The curves $\mathbf{x} = \boldsymbol{\phi}(x_0, t)$ $\mathbf{x}_0$ fixed, are the *paths of motion* of the flow. The velocity $\mathbf{v}(\mathbf{x}, t)$ of the particle at $\mathbf{x}$ at time t is the *velocity field* of the flow

$$\mathbf{v}(\mathbf{x}, t) = \frac{\partial \boldsymbol{\phi}}{\partial t}(\mathbf{x}_0, t)\Big|_{\mathbf{x} = \boldsymbol{\phi}(\mathbf{x}_0, t)}$$

If $\mathbf{v}$ is independent of t, the flow is *steady*. The velocity field of a flow completely determines the flow, for the paths of motion are obtained by solving the differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t)$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

When the flow is steady the paths of motion do not change with time, and particles on the same path remain on the same path.

• FURTHER READING

In addition to the bibliography at the end of Chapter 3, we should also mention these excellent texts on differential geometry:

D. Struik, *Lectures on Classical Differential Geometry*, Addison-Wesley, Reading, Mass., 1950.

H. Guggenheimer, *Differential Geometry*, McGraw Hill, New York, 1963.

R. T. Seeley, *Calculus*, Scott-Foresman, Glenview, Ill., 1967 has a derivation of Newton's law of gravitational attraction from Kepler's laws.

• MISCELLANEOUS PROBLEMS

48. Suppose that γ is a closed curve in the plane which lies outside the unit disk and encircles the origin. Show that the length of γ is at least 2π.

49. Suppose that γ is a closed curve lying completely inside the unit disk with the property that it crosses every ray once and only once. Is there an a priori bound on the length of γ?

50. Suppose that γ is a curve as described in Problem 49, whose curvature is bounded by 1. Is there now a bound to the length of γ?

51. A pendulum consists of a body of mass m hanging on a rope of length L which is fixed at one end. If the mass is displaced from the vertical and let go it will swing along the circle of radius L. Find the differential equation of the motion.

52. Suppose a particle is moving along the curve of Example 20 at constant speed. Find the speed of its projection onto the xy plane.

53. Suppose that a particle moves along the right circular cone according to the equation

$$\mathbf{x} = (\cos t, \sin t, 2t)$$

Find the equation of motion of the projection of the particle on the plane $x = 1$.

54. A horse is running around the elliptical track

$$x^2 + 2y^2 = 1$$

at constant speed. There is a wall along the line $y = -1$ and a floodlight at the point (0, 1) which casts the horse's shadow on the wall. Find the equation of motion of the shadow.

55. A man six feet tall walks at constant speed along a straight line passing directly beneath a street lamp 12 feet off the ground. Find the equation of motion of the head of the man's shadow cast by the street lamp.

56. A loose foot bridge of length L hangs across a chasm of width W $(L > W)$. A man appears at one entrance on a pair of roller skates.

Suddenly he lets go and begins skating down the bridge. Assuming the only forces acting on him are those due to gravity and the restraining forces of the bridge, find the differential equation governing his motion.

57. Why does a river going around a curve wear out the far bank and deposit silt along the near bank directly after the curve?

58. Suppose a disk of radius r rolls with constant speed (at the center) along a disk of radius R in the plane. Find the equation of motion of a typical point on the circumference of the smaller disk.

59. Find the differential equation of the motion of a ball rolling in a parabolic dish (with profile $y = x^2$) starting at rest at some point other than the center.

60. Assuming that the population of the organisms on a given remote island remains bounded, can you say anything about the eigenvalues of the biotic matrix?

61. Find the curvature and torsion of these curves in R^3:

(a) $\mathbf{x} = (u - \sin u, 1 - \cos u, 3u)$

(b) $\mathbf{x} = (\sin u, 1 + \cos u, \sin u)$

62. Let $\mathbf{x} = \mathbf{x}(s)$ be the equation of a curve γ in R^3 whose tangent vector $\mathbf{T}(s)$ traces out a circle on the sphere. Show that γ is a helix.

63. A *general helix* is a curve lying on a surface of revolution $z = f(r)$ which cuts the curves $z = f(r)$, $\theta =$ constant at a fixed angle. Show that the ratio κ/τ is constant on a helix.

64. Find the curve on the xy plane onto which a helix on a cone projects.

65. Let γ_1 and γ_2 be two space curves for which we have a point for point correspondence such that the line joining corresponding points is the normal line to both curves. Show that the line segment between corresponding points has constant length.

66. Let $\mathbf{x} = \mathbf{x}(t)$ be an R^n-valued function of a real variable which is n-times continuously differentiable. Then the image of γ is a curve in R^n. The *Frenet-Serret frame* of γ is the orthonormal set obtained by applying the Gram-Schmidt process to the vectors

$$\mathbf{x}'(t), \mathbf{x}''(t), \ldots, \mathbf{x}^{(n)}(t)$$

(a) Show that for $n = 3$, the Frenet-Serret frame is $\mathbf{T}$, $\pm\mathbf{N}$, $\pm\mathbf{B}$.

(b) Show that if there are only k independent vectors in the Frenet-Serret frame at every point, the curve lies in a linear subspace of dimension k.

(c) Suppose that the Frenet-Serret frame $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n$ is a basis. Show that the matrix representing the vectors $d\mathbf{T}_1/ds, \ldots, d\mathbf{T}_n/ds$ in this basis is skew-symmetric. These are the generalized Frenet-Serret formulas.

67. Find the Frenet-Serret formulas for the curve

$$\mathbf{x} = (\cos t, \sin t, t, 2t)$$

in R^4.

68. Kepler's laws of planetary motion (from which Newton derived his law of gravitational attraction) are these:

I. For each planet the ray from the sun to the planet sweeps out equal areas in equal times.

II. The path of motion of each planet is an ellipse with the sun at one focus.

III. The square of the time period required to make one revolution is proportional to the cube of the major axis of the ellipse. This constant of proportionality is the same for all the planets.

In the text we have derived Kepler's second law from Newton's laws. Now derive Kepler's first and third laws.